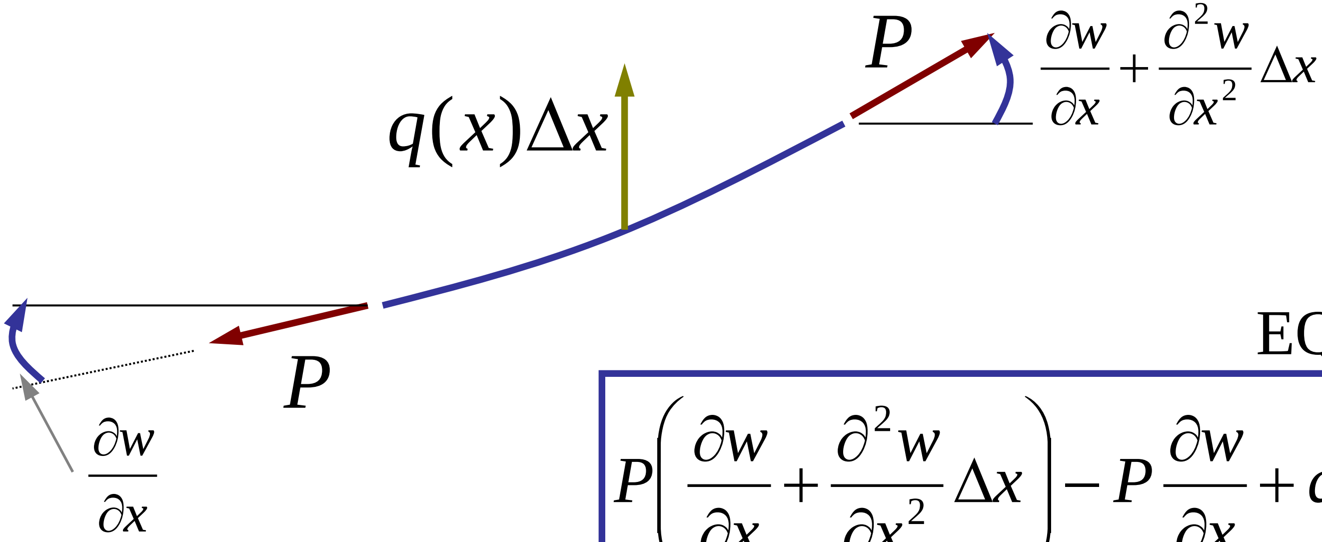
TWO-DIMENSIONAL FINITE ELEMENT ANALYSIS

- THE MEMBRANE PROBLEM
- MESHING IN 2D
- BASIS/SHAPE FUNCTIONS
- MASTER ELEMENT + ELEMENT CALCULATIONS
- AN EXAMPLE

STRINGS….



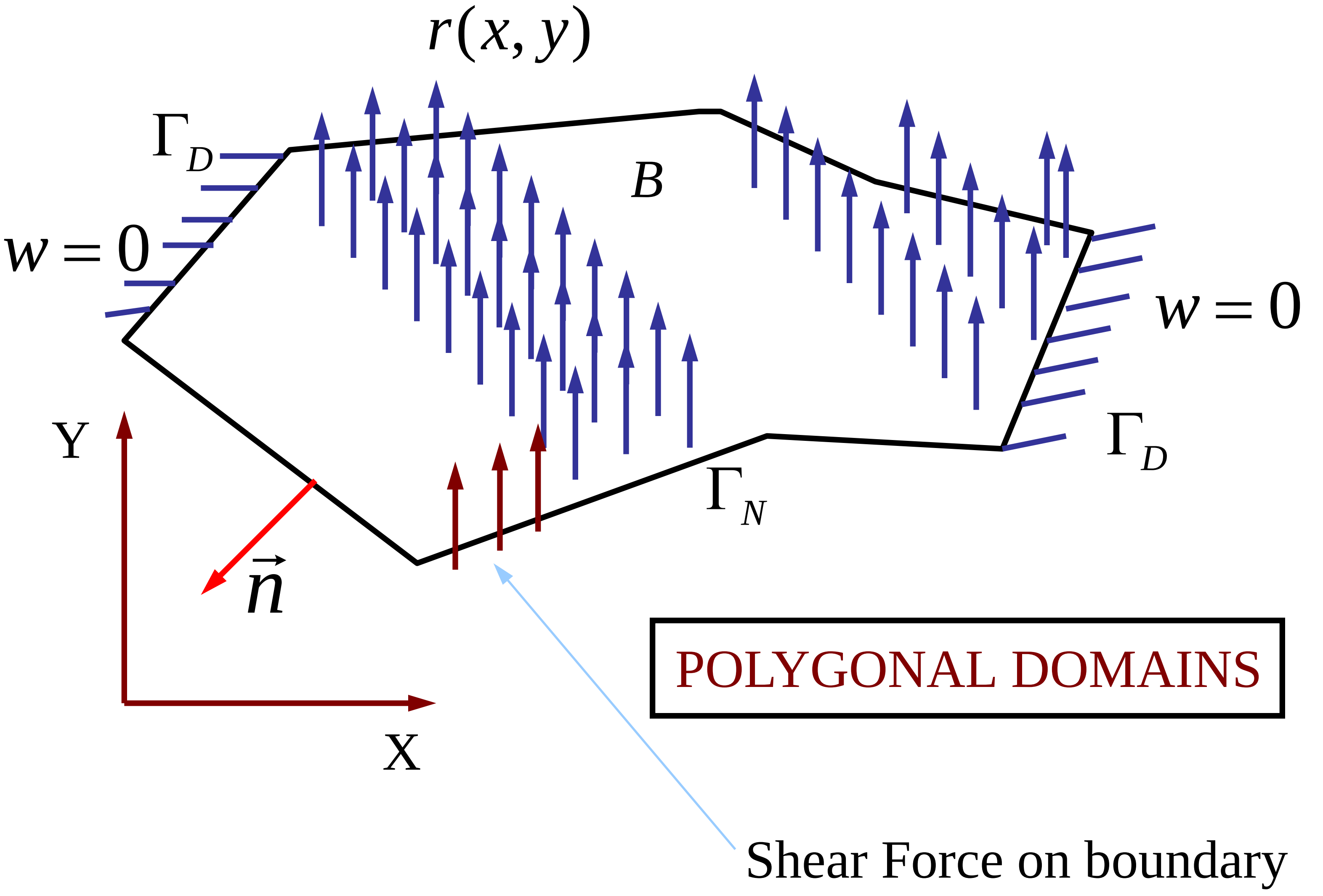
EQUILIBRIUM

$$P\left(\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}\Delta x\right)-P\frac{\partial w}{\partial x}+q\Delta x=0$$

$$\Rightarrow\ -P\frac{\partial^2 w}{\partial x^2}=q\quad\Rightarrow\quad -\frac{\partial^2 w}{\partial x^2}=\frac{q}{P}=r(x)$$

MEMBRANES

$$-\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=r(x,y)$$



POLYGONAL DOMAINS

$$\int_B \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} \right) dA = \int_B (rv)\, dA + \int_{\Gamma_D+\Gamma_N} \underbrace{\left(\frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \right)}_{\frac{\partial w}{\partial n}} v\, dS$$

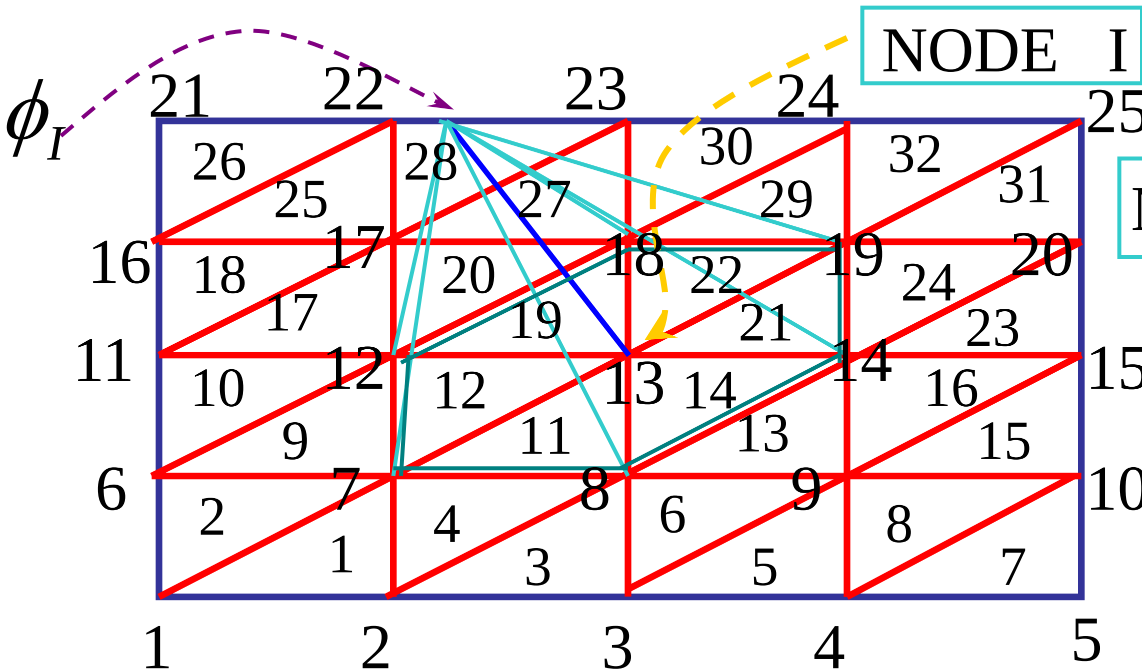
$$= \int_{\Gamma_N} \frac{\partial w}{\partial n} v\, dS$$

THE WEAK FORMULATION

$$v = 0 \quad \text{on} \quad \Gamma_D$$

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_N} = g(x(s), y(s))$$

- NEED C^0 ELEMENTS IN 2D.
- MESH HAS TO BE FIRST MADE, THEN THE BASIS FUNCTIONS DESIGNED.



MESH OF TRIANGLES

$$\phi_i(x, y) = a_0 + a_1 x + a_2 y$$

PIECEWISE

$$w_{FE}(x, y) = \sum_{i=1}^{25} \alpha_i \phi_i(x, y)$$

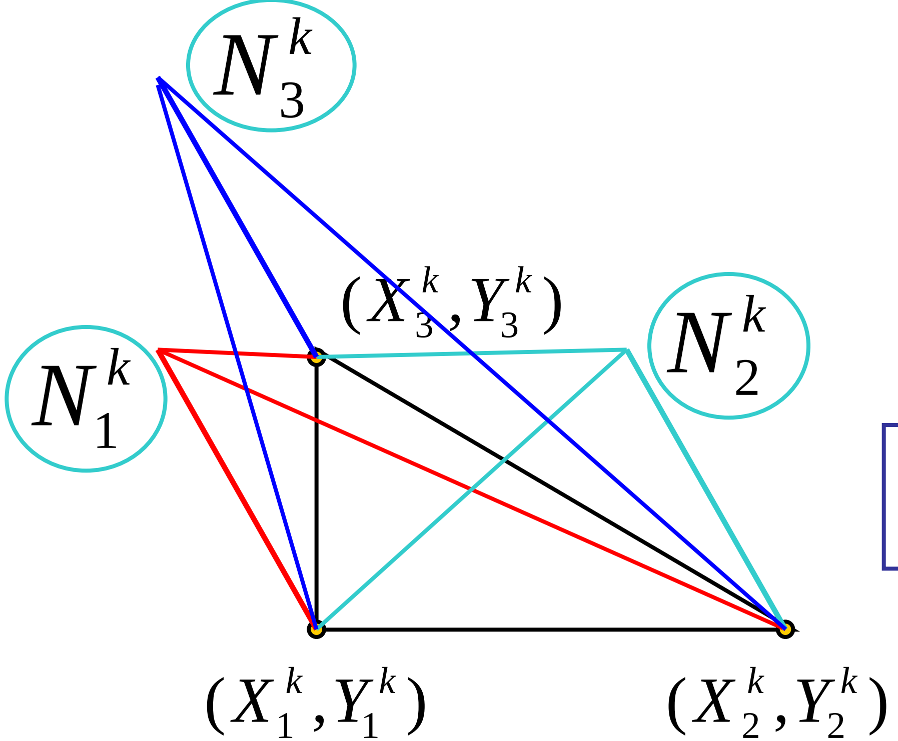
THE i th EQUATION

$$\int_B \left(\frac{\partial w_{FE}}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{\partial w_{FE}}{\partial y}\frac{\partial \phi_i}{\partial y} \right) dA = \int_B (r\phi_i)\, dA + \int_{\Gamma_N} \left(\frac{\partial w}{\partial n} \right) \phi_i \, dS$$

$$\sum_{k=1}^{NEL} \int_{I_k} \left(\frac{\partial w_{FE}}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{\partial w_{FE}}{\partial y}\frac{\partial \phi_i}{\partial y} \right) dA = \sum_{k=1}^{NEL} \left(\int_{I_k} (r\phi_i)\, dA + \int_{\partial I_k \cap \Gamma_N} \left(\frac{\partial w}{\partial n} \right) \phi_i \, dS \right)$$

INTEGRALS OVER ELEMENT AREA

ELEMENT EDGE INTEGRAL



ELEMENT k

$$N_1^k(x,y) = \quad 1 \text{ at } x = X_1^k \; ; \; 0 \text{ at } x = X_2^k \; ; 0 \text{ at } x = X_3^k$$

$$\Downarrow$$

$$N_1^k(x,y) = \frac{1}{2A}\left(d_1 + xb_1 + ya_1\right)$$

$$N_2^k(x,y) = \frac{1}{2A}\left(d_2 + xb_2 + ya_2\right)$$

$$N_3^k(x,y) = \frac{1}{2A}\left(d_3 + xb_3 + ya_3\right)$$

$$\bar{a}_i = X_k - X_j$$

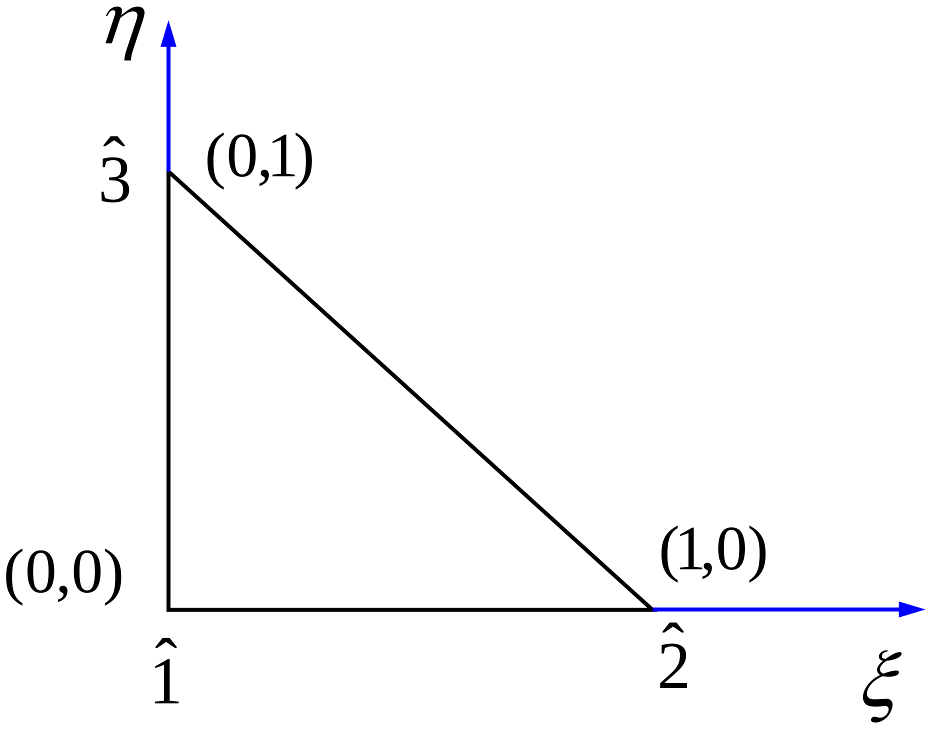
$$\bar{b}_i = Y_k - Y_j$$

$$A = \frac{1}{2}\left(\bar{a}_k \bar{b}_j - \bar{a}_j \bar{b}_k\right)$$

$$d_j = X_k Y_i - X_i Y_j$$

THIS IS A LONG EXPRESSION !!!!

LET US INTRODUCE THE MASTER ELEMENT



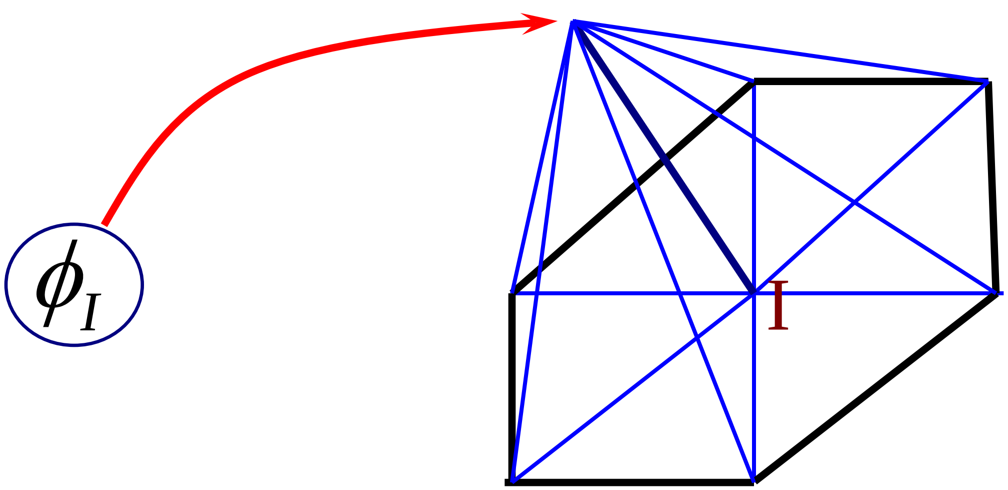
$$x = X_1^k(1-\xi-\eta) + X_2^k\,\xi + X_3^k\,\eta$$
$$y = Y_1^k(1-\xi-\eta) + Y_2^k\,\xi + Y_3^k\,\eta$$

LINEAR OR AFFINE MAP

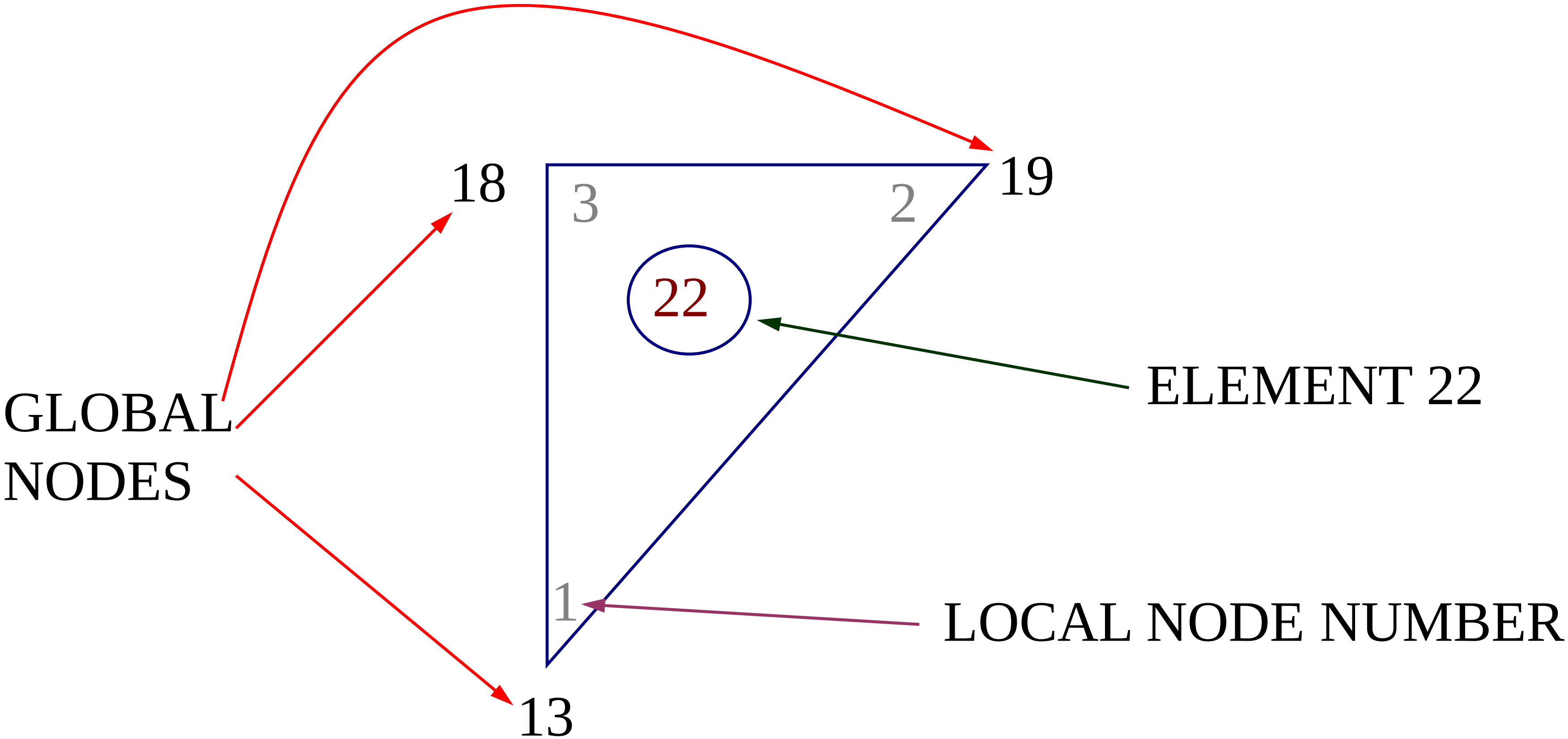
$$\hat{N}_1(\xi,\eta) = 1-\xi-\eta$$

$$\hat{N}_2(\xi,\eta) = \xi$$

$$\hat{N}_3(\xi,\eta) = \eta$$



NOW THE BASIS FUNCTION IS NON-ZERO IN MORE THAN 2 ELEMENTS (6 HERE)



ELEMENT CONNECTIVITY INFORMATION

LOCAL GLOBAL

$1 \rightarrow 13$ $2 \rightarrow 19$ $3 \rightarrow 18$

REPRESENTATION OF FEM SOLUTION IN A GENERIC ELEMENT I_k

$$w_{FE}(x,y)|_{I_k} = \sum_{i=1}^{3} \overline{\alpha}_i^k N_i^k(x,y)$$

$$w_{FE}(x,y)|_{I_{22}} = \sum_{i=1}^{3} \overline{\alpha}_i^k N_i^k(x,y)$$

$$= \alpha_{13}\phi_{13}(x,y) + \alpha_{19}\phi_{19}(x,y) + \alpha_{18}\phi_{18}(x,y)$$

USING THE TRANSFORMATION WE GET

$$\hat{w}_{FE}(x(\xi,\eta), y(\xi,\eta))|_{I_k} = \sum_{i=1}^{3} \overline{\alpha}_i^k \hat{N}_i(\xi,\eta)$$

$$\frac{\partial N_i^k}{\partial x} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i^k}{\partial y} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

CONVERSION OF DERIVATIVES TO MASTER ELEMENT

NEED TO GET THESE

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$

$[J]$

JACOBIAN MATRIX

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

$[J]^{-1}$

$$\frac{\partial x}{\partial \xi} = X_2^k - X_1^k \qquad \frac{\partial x}{\partial \eta} = X_3^k - X_1^k$$

$$\frac{\partial y}{\partial \xi} = Y_2^k - Y_1^k \qquad \frac{\partial y}{\partial \eta} = Y_3^k - Y_1^k$$

THIS CAN BE EVALUATED

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

THE JACOBIAN

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta} \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}$$

THIS CAN BE OBTAINED

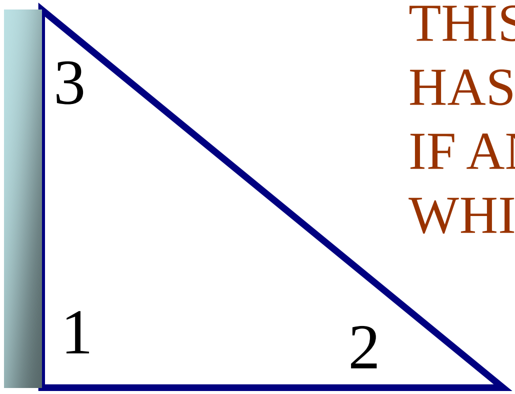
ELEMENT STIFFNESS AND LOAD CALCULATIONS

$$K_{ij}^{(k)} = \int_{\hat{A}} \left(\left(\frac{\partial \hat{N}_j}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_j}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial \hat{N}_j}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_j}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \right) | J | \, d\hat{A}$$

$$F_i^{(k)} = \int_{\hat{A}} \left(\hat{r}(x(\xi,\eta), y(\xi,\eta)) \, \hat{N}_i \right) | J | \, d\hat{A}$$

ON THE BOUNDARY EDGES (NEUMANN EDGE)

$$\frac{\partial w}{\partial n} = g$$



THIS IS EXTRA INFORMATION THAT HAS TO BE CREATED WHILE MESHING. IF AN EDGE LIES ON A BOUNDARY? WHICH BOUNDARY SEGMENT?

$$F_1^{(k)} = F_1^{(k)} + \int_{E_3} \Big(g(s) N_1^k (x(s), y(s)) \Big) ds$$

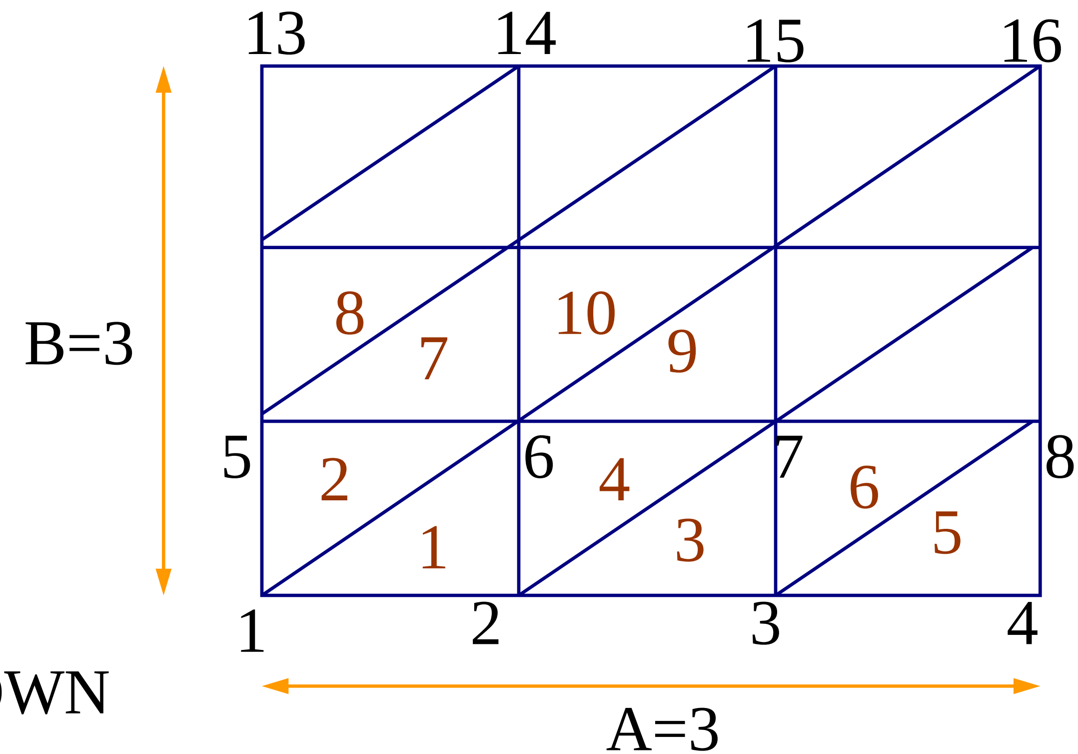
$$F_3^{(k)} = F_3^{(k)} + \int_{E_3} \Big(g(s) N_3^k (x(s), y(s)) \Big) ds$$

AN EXAMPLE

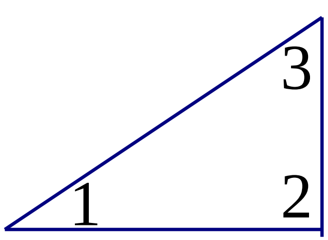
ALL EDGES
CLAMPED

r(x,y)=1

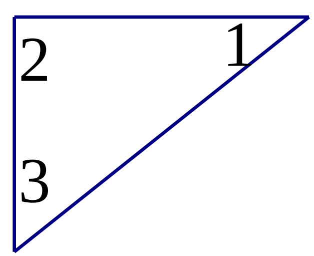
GENERIC
NUMBERING SHOWN



LOCAL
NUMBERING



ELEMENT TYPE 1



ELEMENT TYPE 2

WITH THE GIVEN LOCAL NUMBERING BOTH ELEMENT TYPES HAVE THE SAME STIFFNESS MATRIX AND (IN THIS CASE) LOAD VECTOR

$$[K^{(k)}] = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix}$$

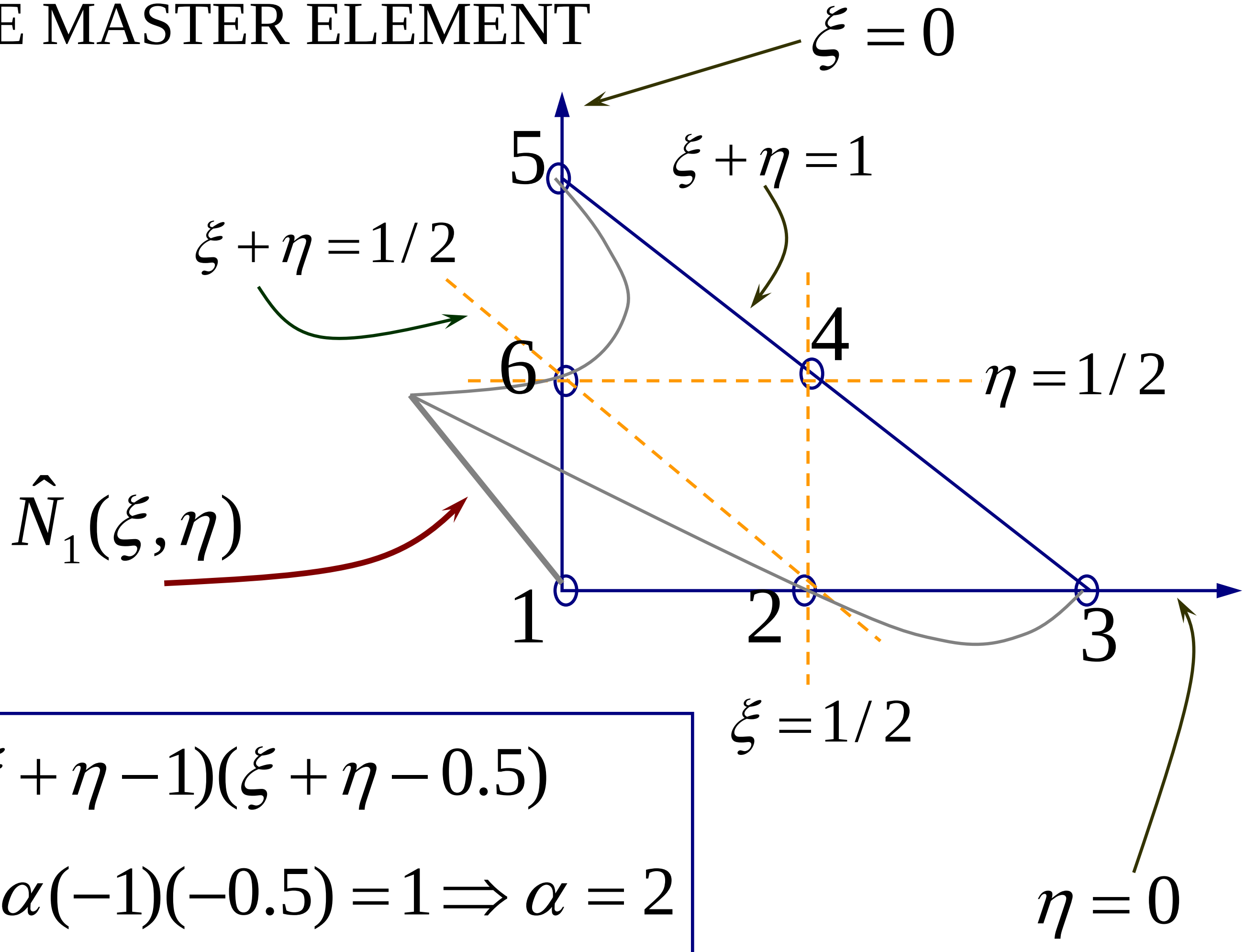
$$\{F^{(k)}\} = \begin{Bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{Bmatrix}$$

NOW THE ELEMENT EQUATIONS CAN BE ASSEMBLED.

THE B.C. REQUIRES ALL DOFS ON THE BOUNDARY TO BE SET TO ZERO – SAME PROCEDURE AS BEFORE.

HIGHER ORDER TRIANGULAR ELEMENTS

WORK IN THE MASTER ELEMENT



$$\hat{N}_1(\xi,\eta) = \alpha(\xi+\eta-1)(\xi+\eta-0.5)$$

$$\hat{N}_1(0,0) = 1 \Rightarrow \alpha(-1)(-0.5) = 1 \Rightarrow \alpha = 2$$

SIMILARLY FOR THE OTHERS. E.g.

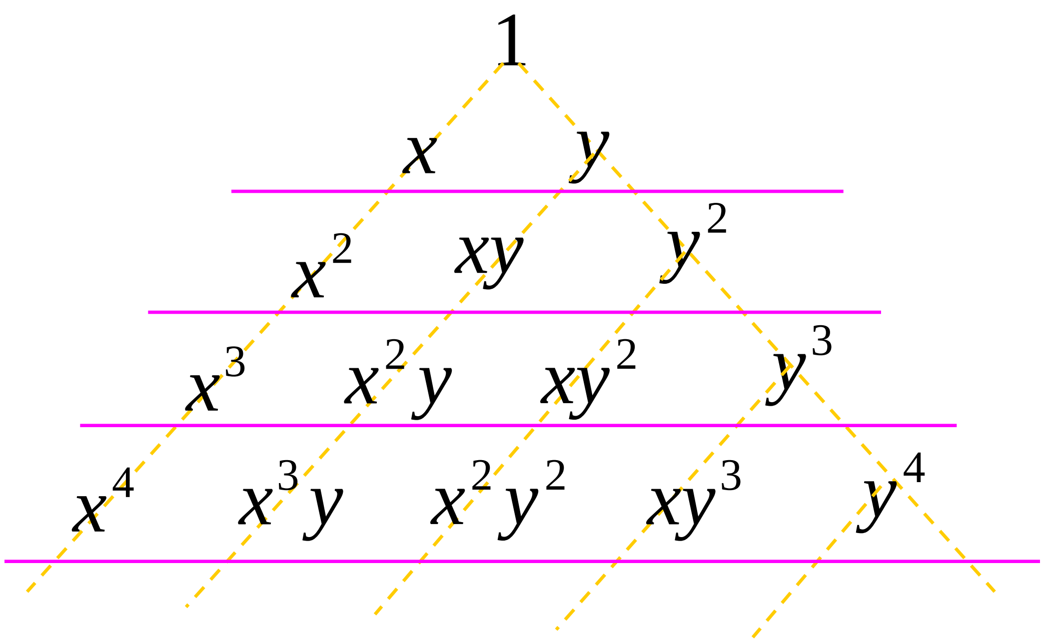
$$\hat{N}_2(\xi,\eta) = \beta(\xi)(\xi+\eta-1)$$

$$\hat{N}_2(0.5,0) = \beta(0.5)(-0.5) = 1 \Rightarrow \beta = -4$$

THERE ARE (p+1)(p+2)/2 SHAPE FUNCTIONS IN GENERAL

PASCAL'S TRIANGLE $\Longrightarrow$ COMPLETENESS

THE PASCAL TRIANGLE



THE CHOSEN BASIS FUNCTIONS OF ORDER p SHOULD BE ABLE TO REPRESENT ALL THE MONOMIALS OF ORDER UPTO p. THIS GIVES THE MINIMUM NUMBER OF BASIS FUNCTIONS TO BE $(p+1)(p+2)/2$

THERE ARE FAMILIES WHICH DO MORE – QUADRILATERAL ELEMENTS