TWO-DIMENSIONAL FINITE ELEMENT ANALYSIS

- OTHER ELEMENT TYPES
- PLANAR ELASTICITY PROBLEM
- DISCUSSION ON ELEMENT CALCULATIONS
- BOUNDARY-CONDITION IMPOSITION
- AN EXAMPLE

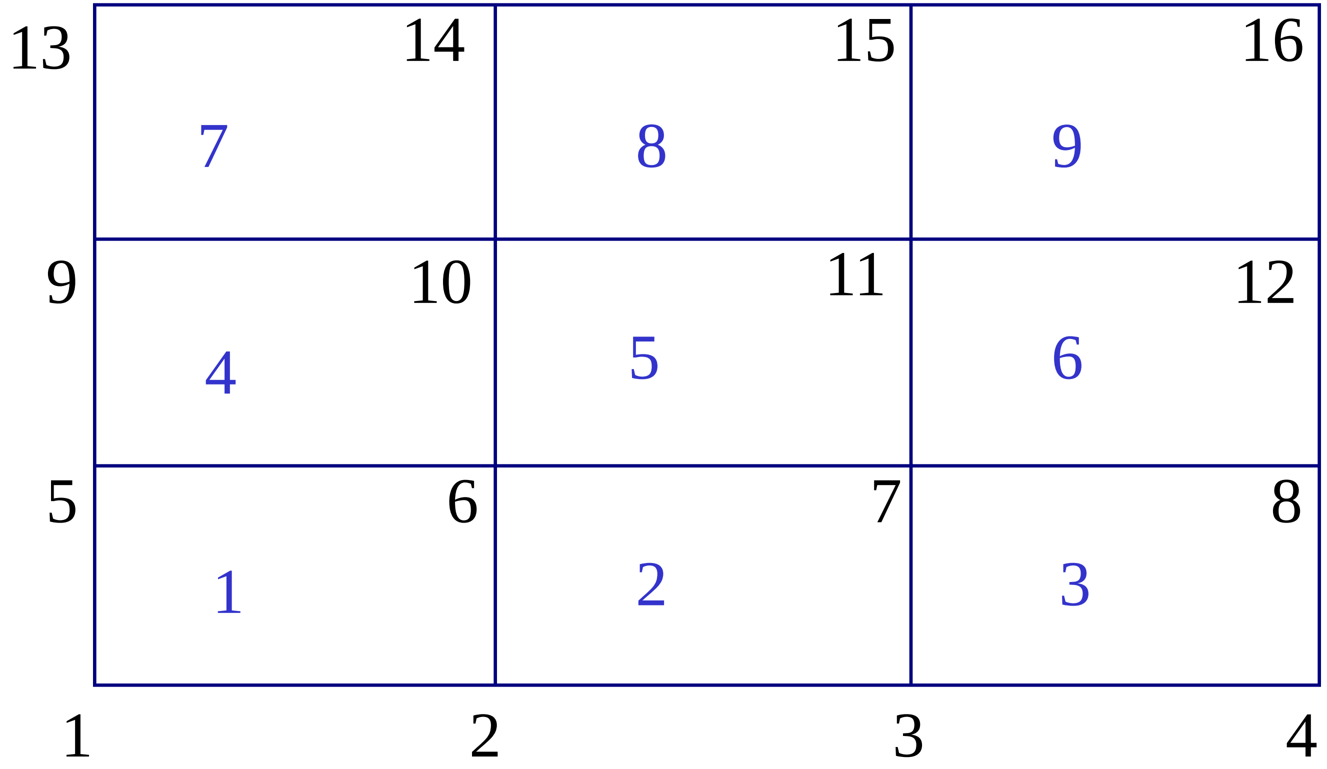
$$\int_B \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} \right) dA = \int_B (rv)\, dA + \int_{\Gamma_D + \Gamma_N} \underbrace{\left(\frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \right)}_{\frac{\partial w}{\partial n}} v\, dS$$

$$= \int_{\Gamma_N} \frac{\partial w}{\partial n} v\, dS$$

THE WEAK FORMULATION

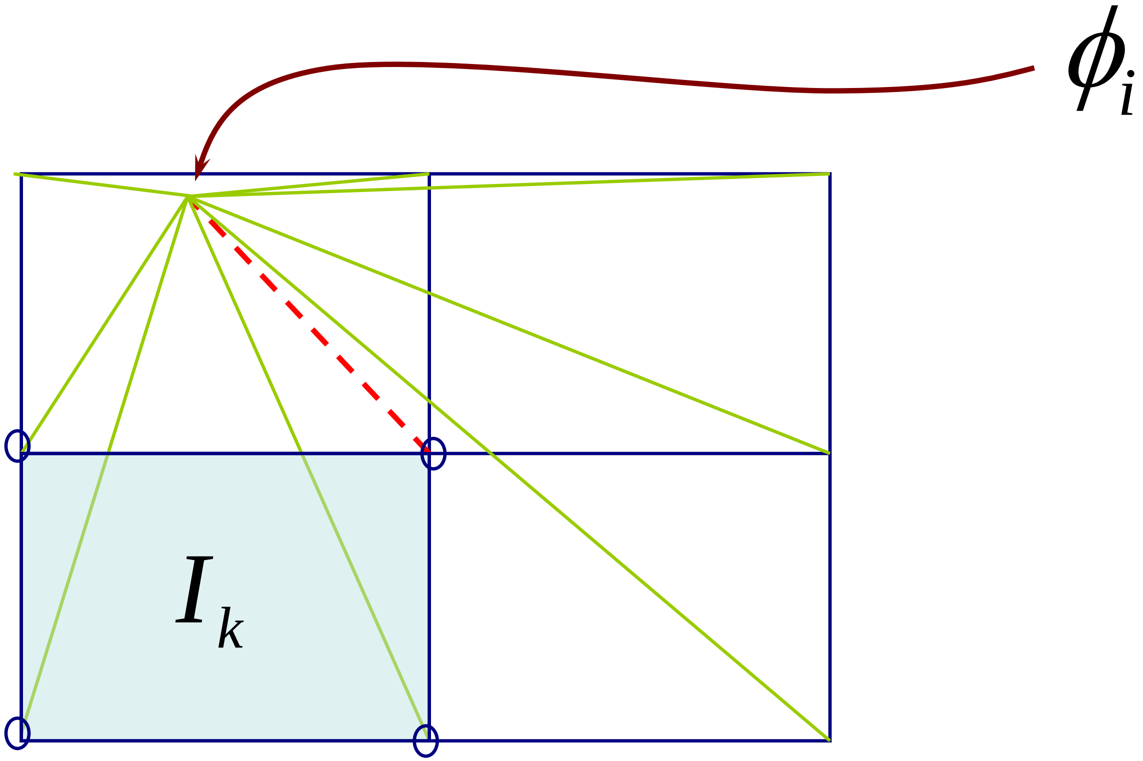
$$v = 0 \quad \text{on} \quad \Gamma_D$$

$$\frac{\partial w}{\partial n}\Big|_{\Gamma_N} = g(x(s), y(s))$$

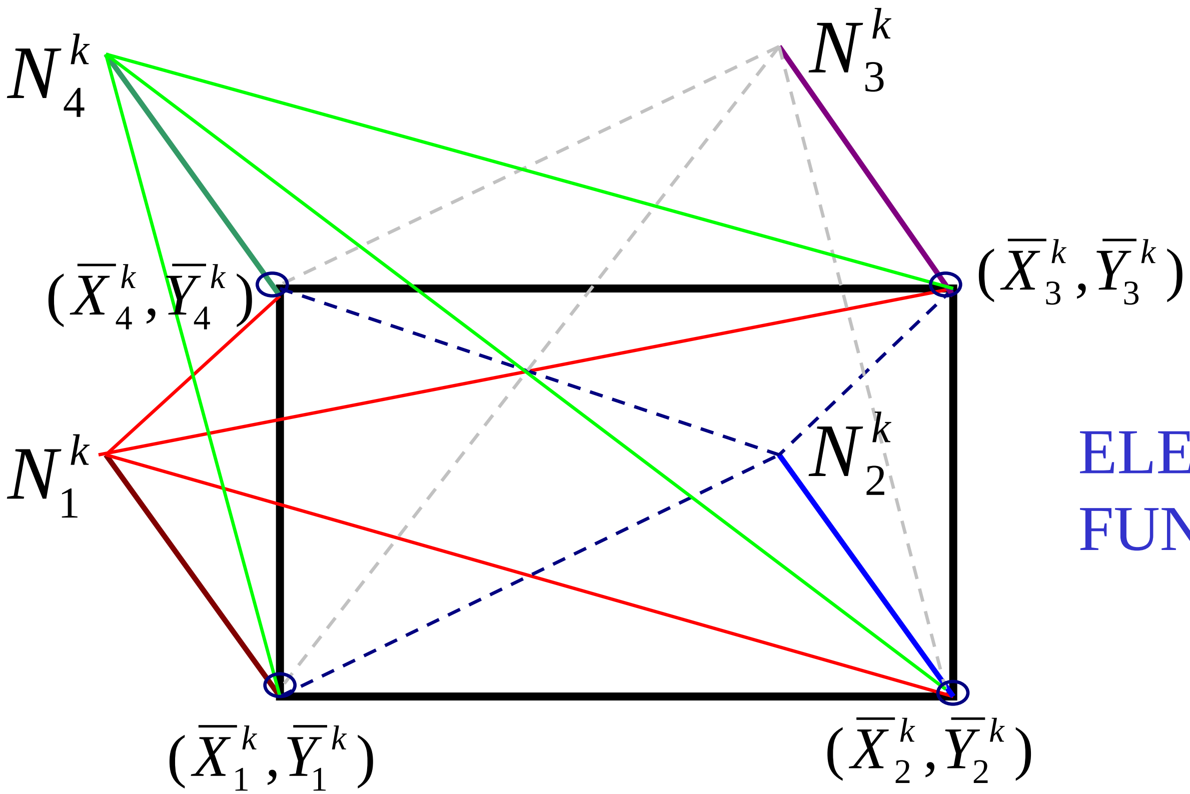


$$w_{FE}(x, y) = \sum_{i=1}^{16} \alpha_i \phi_i(x, y)$$

MESH OF RECTANGLES
(QUADRILATERALS)

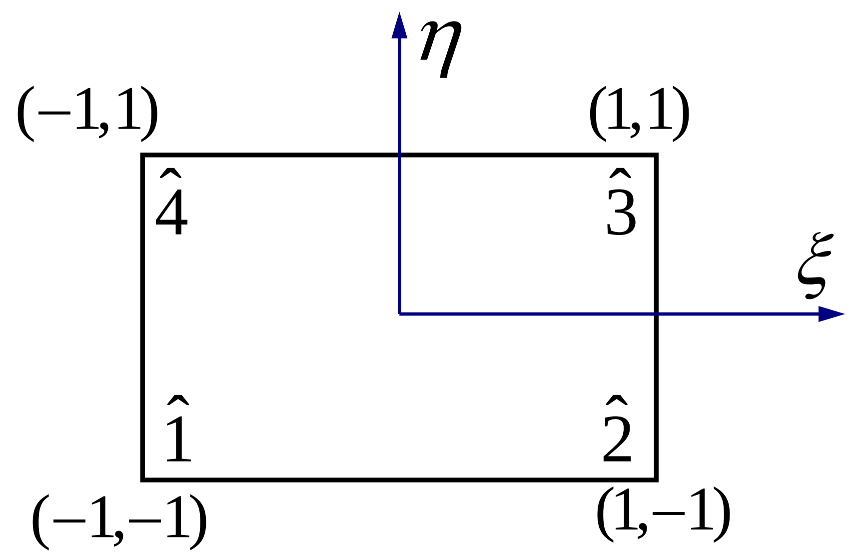


THE REGION WHERE BASIS FUNCTION ϕ_i IS NON-ZERO



ELEMENT SHAPE FUNCTIONS

MASTER ELEMENT



WE ARE GOING TO THE MASTER ELEMENT RIGHT AWAY

$$x(\xi,\eta) = (a_1 + a_2\xi)(b_1 + b_2\eta)$$

BILINEAR MAP

$$x(-1,-1) = \bar{X}_1^k \qquad x(1,-1) = \bar{X}_2^k$$

$$x(1,1) = \bar{X}_3^k \qquad x(-1,1) = \bar{X}_4^k$$

SIMILARLY FOR y

THIS GIVES

$$x(\xi,\eta) = \bar{X}_1^k \frac{(1-\xi)(1-\eta)}{4} + \bar{X}_2^k \frac{(1+\xi)(1-\eta)}{4} + \bar{X}_3^k \frac{(1+\xi)(1+\eta)}{4} + \bar{X}_4^k \frac{(1-\xi)(1+\eta)}{4}$$

$$y(\xi,\eta) = \bar{Y}_1^k \frac{(1-\xi)(1-\eta)}{4} + \bar{Y}_2^k \frac{(1+\xi)(1-\eta)}{4} + \bar{Y}_3^k \frac{(1+\xi)(1+\eta)}{4} + \bar{Y}_4^k \frac{(1-\xi)(1+\eta)}{4}$$

IN THE MASTER ELEMENT, THE SHAPE FUNCTIONS BECOME

$$N_1^k(x,y) = \hat{N}_1(\xi,\eta) = \frac{(1-\xi)(1-\eta)}{4}$$

$$N_2^k(x,y) = \hat{N}_2(\xi,\eta) = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3^k(x,y) = \hat{N}_3(\xi,\eta) = \frac{(1+\xi)(1+\eta)}{4}$$

$$N_4^k(x,y) = \hat{N}_4(\xi,\eta) = \frac{(1-\xi)(1+\eta)}{4}$$

BILINEAR SHAPE FUNCTIONS

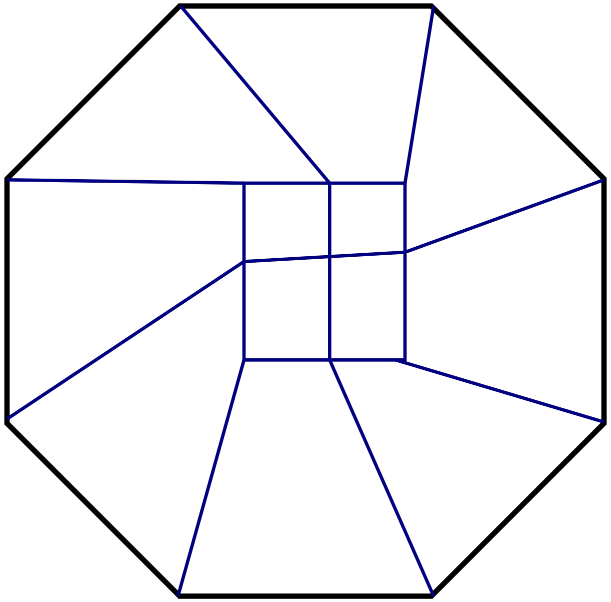
⇓

TENSOR PRODUCT FAMILY OF SHAPE FUNCTIONS and

SERENDIPITY FAMILY OF SHAPE FUNCTIONS

PRODUCT OF 1D-SHAPE FUNCTIONS IN THE TWO DIRECTIONS

NOW THESE CAN BE USED FOR MESHES OF QUADRILATERALS ALSO



THE PROCEDURE FOR OBTAINING THE METRICS OF THE TRANSFORMATION REMAINS EXACTLY THE SAME

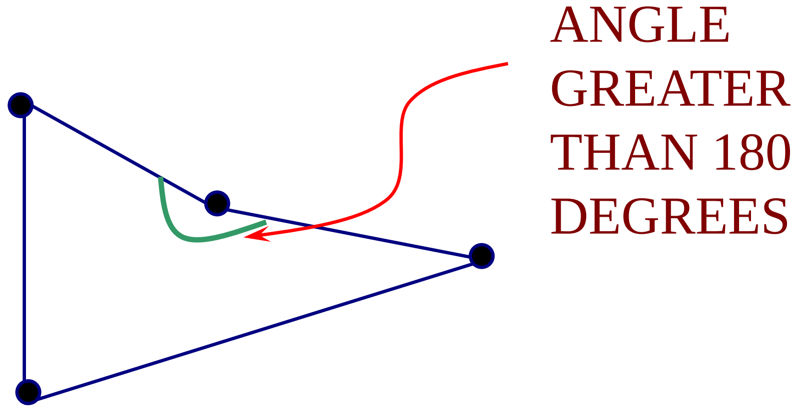
$$\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \xi}, \frac{\partial y}{\partial \eta}$$

$$\Downarrow$$

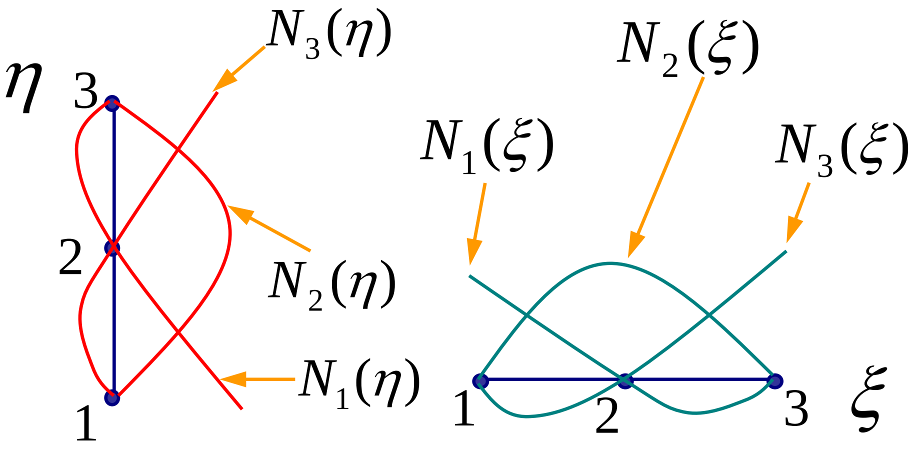
$$\left|J\right| \Rightarrow \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$$

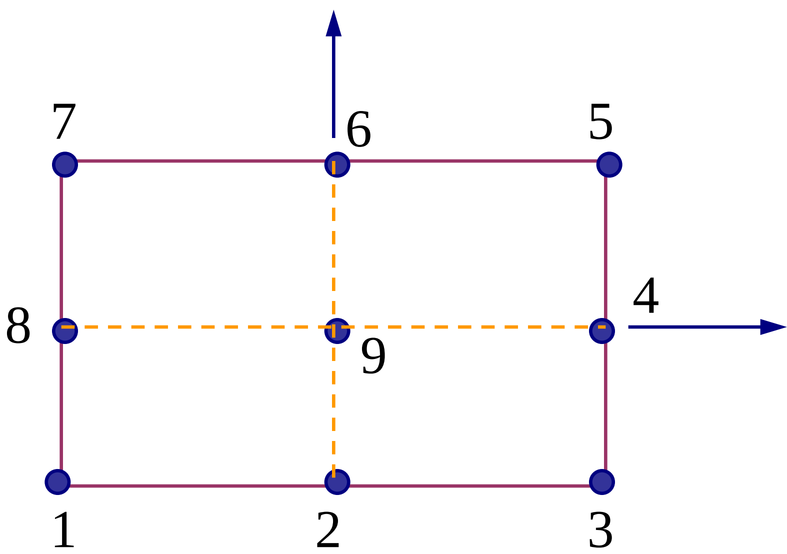
THE JACOBIAN IS NOW A FUNCTION OF ξ, η AND NOT A CONSTANT ANYMORE. HENCE, THE JACOBIAN HAS TO BE EVALUATED AT EACH INTEGRATION POINT

CERTAIN ELEMENT GEOMETRIES SHOULD BE AVOIDED (JACOBIAN BECOMES NEGATIVE OR ZERO). THESE ARE



BI-QUADRATIC ELEMENTS (9 NODED)





$$\hat{N}_1(\xi,\eta) = N_1(\xi)N_1(\eta) \qquad \hat{N}_2(\xi,\eta) = N_2(\xi)N_1(\eta)$$

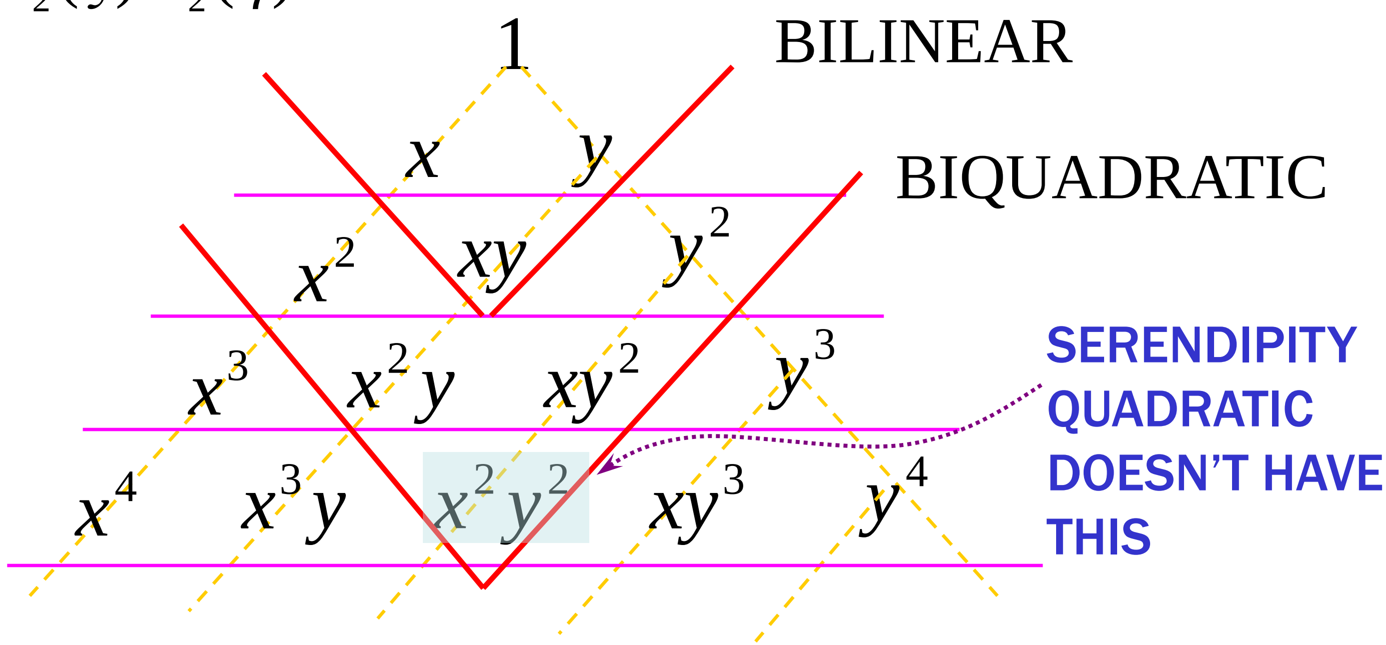
$$\hat{N}_3(\xi,\eta) = N_3(\xi)N_1(\eta) \qquad \hat{N}_4(\xi,\eta) = N_3(\xi)N_2(\eta)$$

$$\hat{N}_5(\xi,\eta) = N_3(\xi)N_3(\eta) \qquad \hat{N}_6(\xi,\eta) = N_2(\xi)N_3(\eta)$$

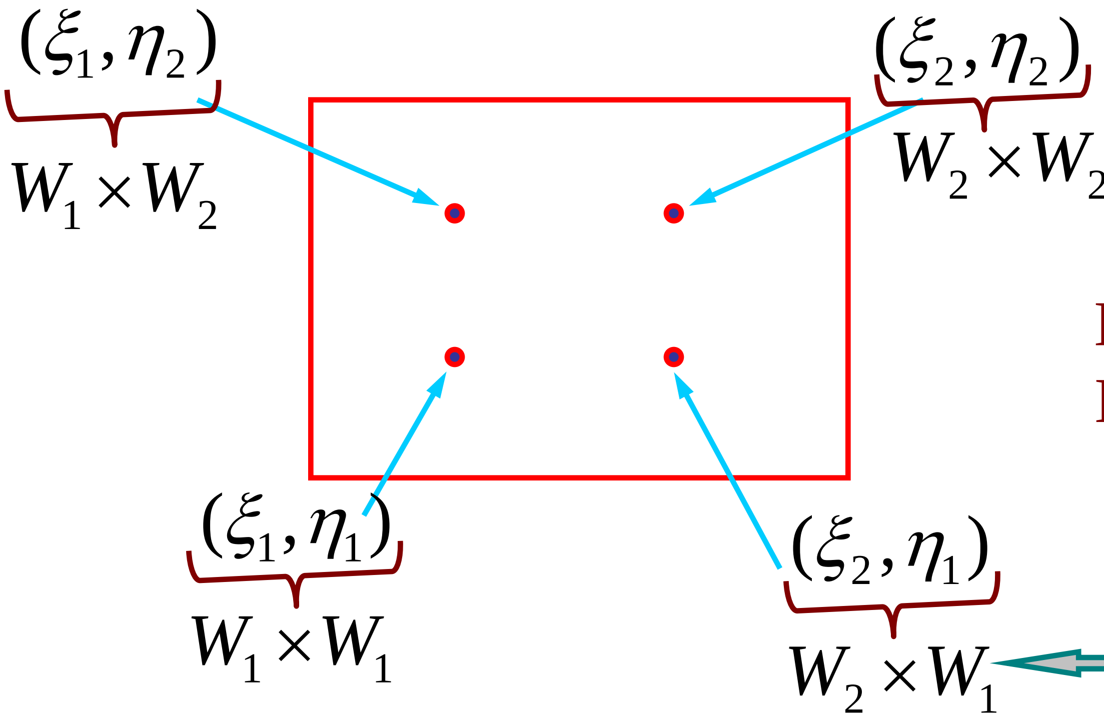
$$\hat{N}_7(\xi,\eta) = N_1(\xi)N_3(\eta) \qquad \hat{N}_8(\xi,\eta) = N_1(\xi)N_2(\eta)$$

$$\hat{N}_9(\xi,\eta) = N_2(\xi)N_2(\eta)$$

MORE THAN COMPLETE !

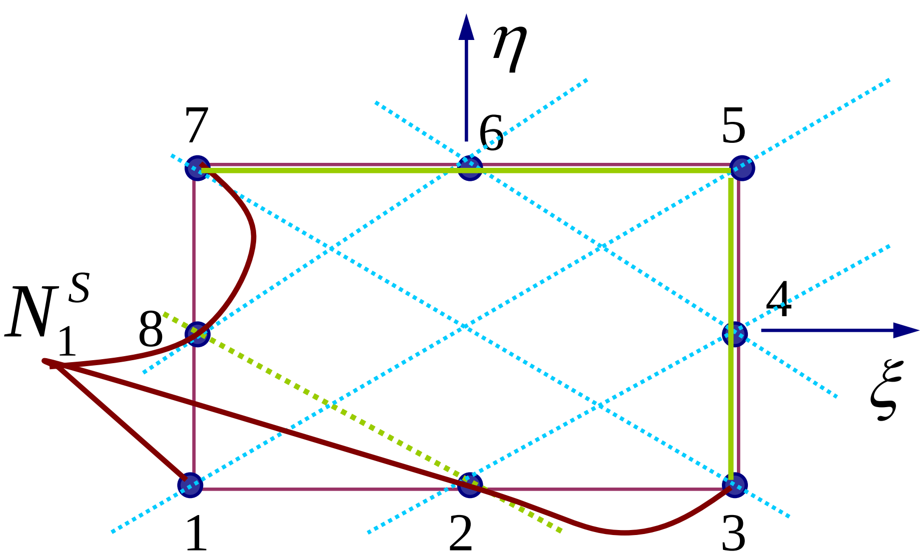


CAN USE 1D INTEGRATION POINT IN BOTH DIRECTIONS



EXAMPLE OF A 2 x 2 INTEGRATION RULE

WEIGHTS ARE ALSO PRODUCTS OF 1D- WEIGHTS



$$N_1^S = \alpha(\xi + \eta + 1)(\xi - 1)(\eta - 1)$$

SERENDIPITY ELEMENTS (8 NODED)

THE i th EQUATION

$$\int_B \left(\frac{\partial w_{FE}}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{\partial w_{FE}}{\partial y}\frac{\partial \phi_i}{\partial y} \right) dA = \int_B (r\phi_i)\, dA + \int_{\Gamma_N} \left(\frac{\partial w}{\partial n} \right) \phi_i \, dS$$

$$\sum_{k=1}^{NEL} \int_{I_k} \left(\frac{\partial w_{FE}}{\partial x}\frac{\partial \phi_i}{\partial x} + \frac{\partial w_{FE}}{\partial y}\frac{\partial \phi_i}{\partial y} \right) dA = \sum_{k=1}^{NEL} \left(\int_{I_k} (r\phi_i)\, dA + \int_{\partial I_k \cap \Gamma_N} \left(\frac{\partial w}{\partial n} \right) \phi_i \, dS \right)$$

INTEGRALS OVER ELEMENT AREA

ELEMENT EDGE INTEGRAL

$$\frac{\partial N_i^k}{\partial x} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i^k}{\partial y} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

CONVERSION OF DERIVATIVES TO MASTER ELEMENT

NEED TO GET THESE

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$

$[J]$

JACOBIAN MATRIX

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

$[J]^{-1}$

$$\frac{\partial x}{\partial \xi} \qquad \frac{\partial x}{\partial \eta}$$

$$\frac{\partial y}{\partial \xi} \qquad \frac{\partial y}{\partial \eta}$$

THIS CAN BE EVALUATED

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

THE JACOBIAN

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta} \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}$$

THIS CAN BE OBTAINED

ELEMENT STIFFNESS AND LOAD CALCULATIONS

$$K_{ij}^{(k)} = \int_{\hat{A}} \left(\left(\frac{\partial \hat{N}_j}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_j}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial \hat{N}_j}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_j}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \right) | J | \, d\hat{A}$$

$$F_i^{(k)} = \int_{\hat{A}} \left(\hat{r}(x(\xi,\eta), y(\xi,\eta)) \, \hat{N}_i \right) | J | \, d\hat{A}$$

THE PLANAR ELASTICITY PROBLEM

THE GLOBAL BALANCE LAW

$$\left.\begin{aligned} r_x &\Rightarrow \sigma_{xx,x} + \sigma_{yx,y} + f_x = 0 \\ r_y &\Rightarrow \sigma_{xy,x} + \sigma_{yy,y} + f_y = 0 \end{aligned}\right\} \text{ in } B$$

RESIDUE VECTOR $\{r\}$ $\Rightarrow$ $\displaystyle\int_B \{r\}\cdot\{v\}\, dA = 0$

WEIGHTED RESIDUAL FORM

WEIGHT FUNCTION OR VIRTUAL DISPLACEMENT VECTOR

$$\int_B \left[\left(\sigma_{xx,x} + \sigma_{yx,y} + f_x \right) v_x + \left(\sigma_{xy,x} + \sigma_{yy,y} + f_y \right) v_y \right] dA = 0$$

Integrate by parts

$$\int_B \left[\left(-\sigma_{xx} v_{x,x} - \sigma_{yx} v_{x,y} + f_x v_x \right) + \left(-\sigma_{xy} v_{y,x} - \sigma_{yy} v_{y,y} + f_y v_y \right) \right] dA$$

$$+ \int_{\partial B} \left[\left(\sigma_{xx} n_x + \sigma_{yx} n_y \right) v_x + \left(\sigma_{xy} n_x + \sigma_{yy} n_y \right) v_y \right] = 0$$

T_x T_y

Boundary Traction

WEAK OR VIRTUAL WORK FORMULATION

$$\int_B \left[\left(\sigma_{xx} \varepsilon_{xx}(v) + \sigma_{xy} \gamma_{xy}(v) + \sigma_{yy} \varepsilon_{yy}(v) \right) \right] dA =$$

$$\int_B \left(f_x v_x + f_y v_y \right) dA + \int_{\partial B} \left(T_x v_x + T_y v_y \right) dS$$

GENERALIZED LINEAR ELASTICITY

CONSTITUTIVE RELATION

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma(u)\} \qquad [C]$$

$$\begin{Bmatrix} \varepsilon_{xx}(u) \\ \varepsilon_{yy}(u) \\ \gamma_{xy}(u) \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{Bmatrix}$$

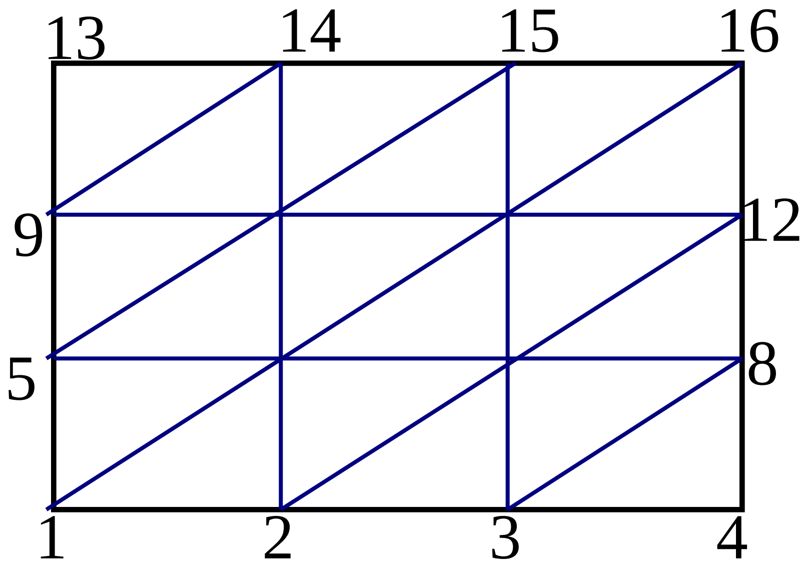
STRAIN-DISPLACEMENT RELATION

$$\{\varepsilon(u)\}$$

THE LEFT HAND SIDE OF THE WEAK FORM

$$\int_B \{\varepsilon(v)\}^T \{\sigma(u)\} \, dA = \int_B \{\varepsilon(v)\}^T [C] \{\varepsilon(u)\} \, dA$$

NEED C^0 BASIS FUNCTIONS FOR BOTH $u_x \, , u_y$



$$u_{x_{FE}} = \sum_{i=1}^{16} \alpha_i \, \phi_i(x, y)$$

$$u_{y_{FE}} = \sum_{i=1}^{16} \beta_i \, \phi_i(x, y)$$

THIS IS EQUIVALENT TO

$$\begin{Bmatrix} u_{x_{FE}} \\ u_{y_{FE}} \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \cdots & \phi_{16} & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & \cdots & 0 & \phi_{16} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \\ \vdots \\ \alpha_{16} \\ \beta_{16} \end{Bmatrix}$$

$$\{u_{FE}\} = [\phi]\{\alpha\}$$

$$\{v_{FE}\} = [\phi]\{\delta\}$$

The virtual displacement

$$\{\varepsilon(u_{FE})\} = \begin{Bmatrix} u_{x_{FE},x} \\ u_{y_{FE},y} \\ u_{x_{FE},y} + u_{y_{FE},x} \end{Bmatrix} = [B]\{\alpha\}$$

$$[B] = \begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \phi_{3,x} & 0 & \cdots \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & 0 & \phi_{3,y} & \cdots \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \phi_{3,y} & \phi_{3,x} & \cdots \end{bmatrix}$$

$$\{\varepsilon(v_{FE})\} = [B]\{\delta\}$$

$$\{\delta\}^T \left[\int_B [B]^T [C][B]\{\alpha\} dA - \int_B [\phi]^T \{f\} dA - \int_{\partial B} [\phi]^T \{T\} dS \right] = 0$$

$$\left[\int_B [B]^T [C][B] dA \right] \{\alpha\} = \int_B [\phi]^T \{f\} dA + \int_{\partial B} [\phi]^T \{T\} dS$$

$[K]$ ⇐ 2*NNDOF X 2*NNDOF sized stiffness matrix

NNDOF = 16 for the example