TWO-DIMENSIONAL FINITE ELEMENT ANALYSIS

- THE PLANAR ELASTICITY PROBLEM REVISITED
- INCLINED BOUNDARIES
- CURVED BOUNDARIES
- CURVED ELEMENTS (SUB, ISO, SUPER PARAMETRIC FORMULATION)
- DISCUSSION ON COMPLETENESS ISSUES
- BOUNDARY-CONDITION IMPOSITION
- MESH DESIGN, CONVERGENCE ISSUES

THE PLANAR ELASTICITY PROBLEM

THE GLOBAL BALANCE LAW

$$r_x \Rightarrow \sigma_{xx,x} + \sigma_{yx,y} + f_x = 0$$

$$r_y \Rightarrow \sigma_{xy,x} + \sigma_{yy,y} + f_y = 0$$

in B

RESIDUE VECTOR $\{r\}$ $\Rightarrow$ $\int_B \{r\} \cdot \{v\} \, dA = 0$

WEIGHTED RESIDUAL FORM

WEIGHT FUNCTION OR VIRTUAL DISPLACEMENT VECTOR

$$\int_B \left[\left(\sigma_{xx,x} + \sigma_{yx,y} + f_x\right) v_x + \left(\sigma_{xy,x} + \sigma_{yy,y} + f_y\right) v_y\right] dA = 0$$

Integrate by parts

$$\int_B \left[\left(-\sigma_{xx} v_{x,x} - \sigma_{yx} v_{x,y} + f_x v_x\right) + \left(-\sigma_{xy} v_{y,x} - \sigma_{yy} v_{y,y} + f_y v_y\right)\right] dA$$

$$+ \int_{\partial B} \left[\left(\sigma_{xx} n_x + \sigma_{yx} n_y\right) v_x + \left(\sigma_{xy} n_x + \sigma_{yy} n_y\right) v_y\right] = 0$$

T_x T_y

Boundary Traction

WEAK OR VIRTUAL WORK FORMULATION

$$\int_B \left[\left(\sigma_{xx}\varepsilon_{xx}(v) + \sigma_{xy}\gamma_{xy}(v) + \sigma_{yy}\varepsilon_{yy}(v)\right)\right] dA =$$

$$\int_B \left(f_x v_x + f_y v_y\right) dA + \int_{\partial B} \left(T_x v_x + T_y v_y\right) dS$$

GENERALIZED LINEAR ELASTICITY

CONSTITUTIVE RELATION

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx}(u) \\ \varepsilon_{yy}(u) \\ \gamma_{xy}(u) \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{Bmatrix}$$

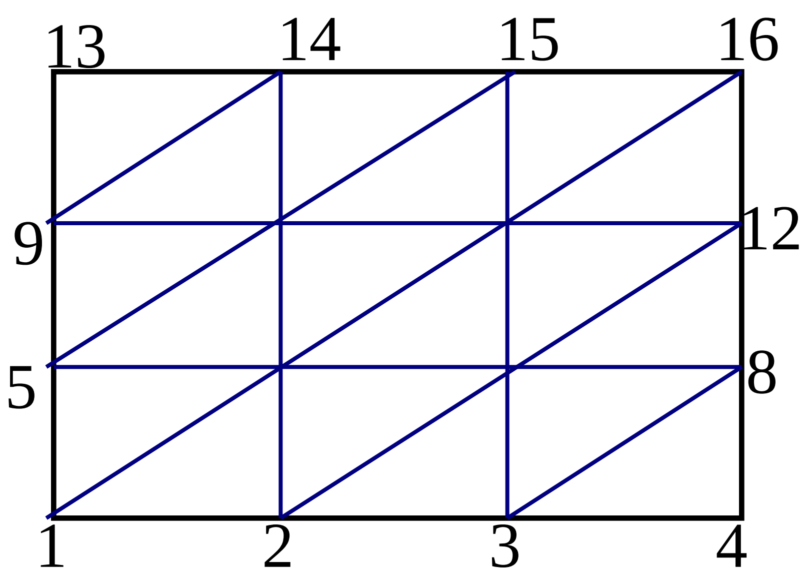
STRAIN-DISPLACEMENT RELATION

$\{\sigma(u)\}$ $\quad [C] \quad$ $\{\varepsilon(u)\}$

THE LEFT HAND SIDE OF THE WEAK FORM

$$\int_B \{\varepsilon(v)\}^T \{\sigma(u)\}\, dA = \int_B \{\varepsilon(v)\}^T [C] \{\varepsilon(u)\}\, dA$$

NEED C^0 BASIS FUNCTIONS FOR BOTH $u_x\ , u_y$



$$u_{x_{FE}} = \sum_{i=1}^{16} \alpha_i\ \phi_i(x,y)$$

$$u_{y_{FE}} = \sum_{i=1}^{16} \beta_i\ \phi_i(x,y)$$

THIS IS EQUIVALENT TO

$$\begin{Bmatrix} u_{x_{FE}} \\ u_{y_{FE}} \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \cdots & \phi_{16} & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & \cdots & 0 & \phi_{16} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \\ \vdots \\ \alpha_{16} \\ \beta_{16} \end{Bmatrix}$$

$$\{u_{FE}\} = [\phi]\{\alpha\}$$

$$\{v_{FE}\} = [\phi]\{\delta\}$$

The virtual displacement

$$\{\varepsilon(u_{FE})\} = \begin{Bmatrix} u_{x_{FE},x} \\ u_{y_{FE},y} \\ u_{x_{FE},y} + u_{y_{FE},x} \end{Bmatrix} = [B]\{\alpha\}$$

$$[B] = \begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \phi_{3,x} & 0 & \cdots \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & 0 & \phi_{3,y} & \cdots \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \phi_{3,y} & \phi_{3,x} & \cdots \end{bmatrix}$$

$$\{\varepsilon(v_{FE})\} = [B]\{\delta\}$$

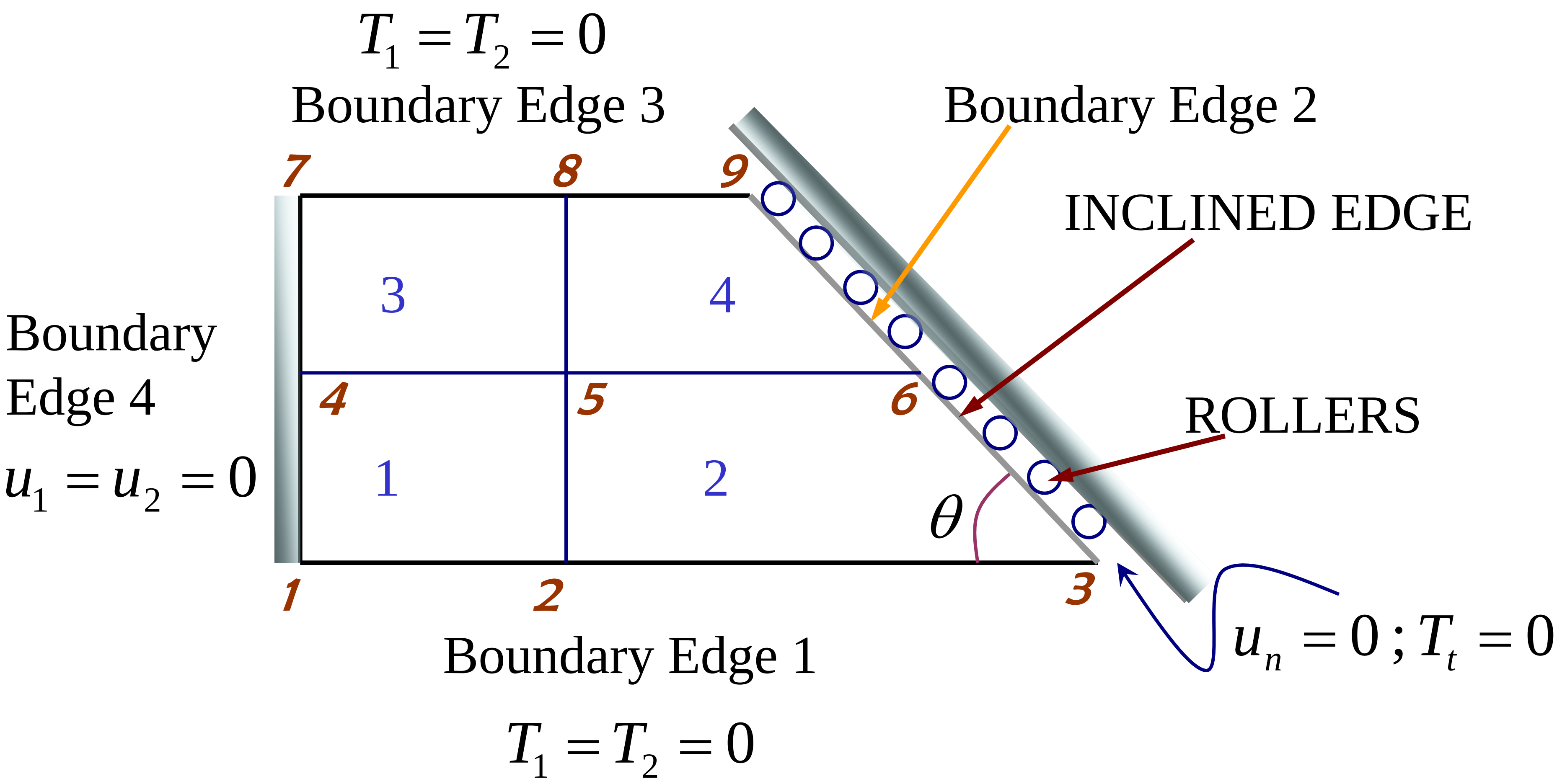
$$\{\delta\}^T \left[\int_B [B]^T [C][B]\{\alpha\}\, dA - \int_B [\phi]^T \{f\}\, dA - \int_{\partial B} [\phi]^T \{T\}\, dS \right] = 0$$

$$\left[\int_B [B]^T [C][B]\, dA \right] \{\alpha\} = \int_B [\phi]^T \{f\}\, dA + \int_{\partial B} [\phi]^T \{T\}\, dS$$

$[K]$ ⇐ 2*NNDOF X 2*NNDOF sized stiffness matrix

NNDOF = 16 for the example

EXAMPLE OF AN INCLINED BOUNDARY



DOFS ON INCLINED EDGE ARE $\alpha_5, \alpha_6, \alpha_{11}, \alpha_{12}, \alpha_{17}, \alpha_{18}$

ON THE FULL INCLINED EDGE WE HAVE:

$$u_n = 0 \;\Rightarrow u_1 \sin\theta + u_2 \cos\theta = 0 \quad (\theta \neq 0{,}90)$$

$$\Rightarrow u_1 = -u_2 \cot\theta$$

THIS IS TRUE FOR EACH NODAL DISPLACEMENT ALSO. HENCE,

$$\alpha_5 = -\alpha_6 \cot\theta;\; \alpha_{11} = -\alpha_{12} \cot\theta; \alpha_{17} = -\alpha_{18} \cot\theta$$

THIS CAN BE WRITTEN AS:

$$[A] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

WITH $c = -\cot\theta$

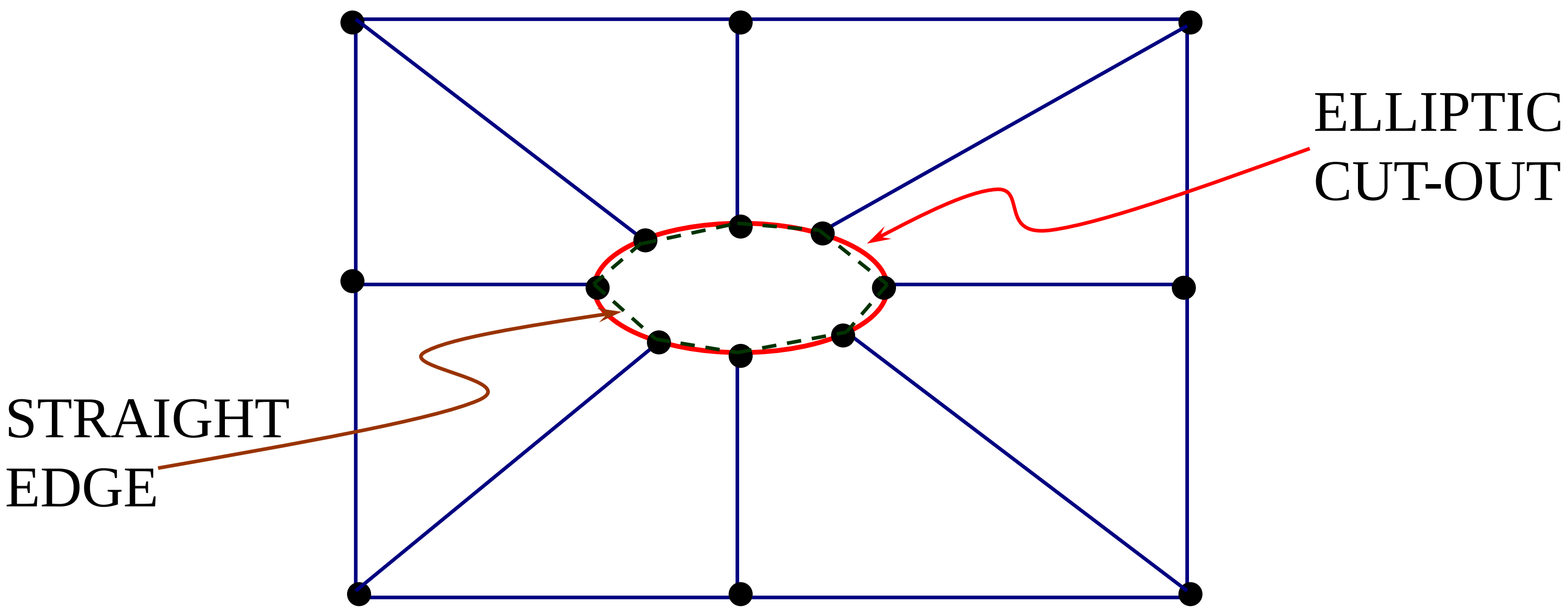
$$\{\alpha\} = [A]\{\overline{\alpha}\}$$

HENCE WE HAVE THE FINAL REDUCED SYSTEM OF EQUATIONS AS

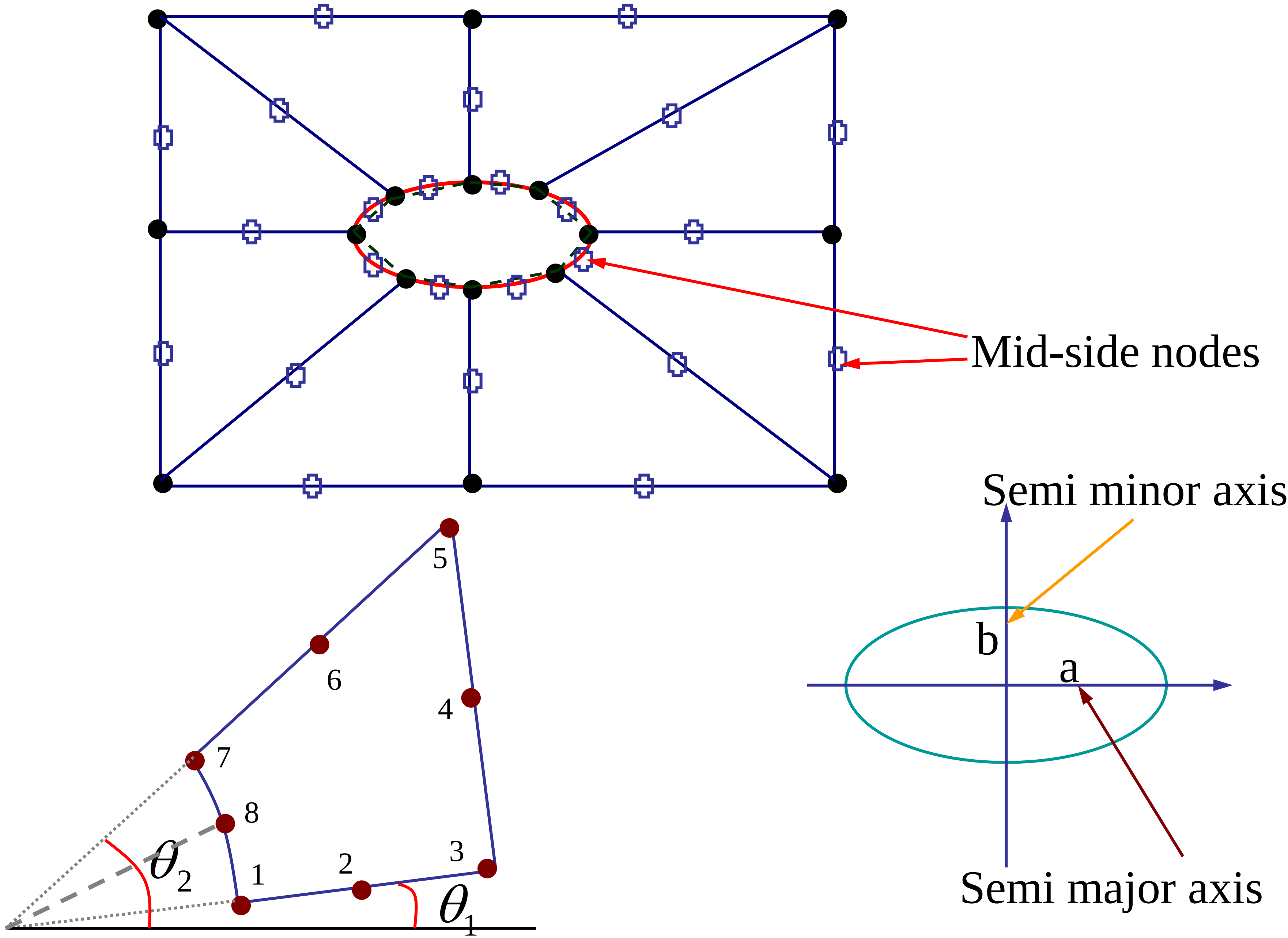
$$[A]^T[K][A]\{\overline{\alpha}\} = [A]^T\{F\}$$

15 UNKNOWNS

DOMAINS WITH CURVED EDGES



LINEAR MAPPING, USING END NODE DATA LEADS TO A MESH THAT *APPROXIMATES THE ELLIPSE WITH A POLYGON*. THIS IS THE DOMAIN THE FEM SOLUTION SEES!



Mid-side nodes
Semi minor axis
b
a
Semi major axis
5
6
4
7
8
3
2
1
θ_2
θ_1

$$X_1^k = a\cos\theta_1\ ;\ X_7^k = a\cos\theta_2 \quad \Rightarrow X_8^k = a\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$Y_1^k = b\sin\theta_1\ ;\ Y_7^k = b\sin\theta_2 \quad \Rightarrow Y_8^k = b\sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$x(\xi,\eta) = \sum_{i=1}^{8} X_i^k \hat{N}_i(\xi,\eta)\ ;\ y(\xi,\eta) = \sum_{i=1}^{8} Y_i^k \hat{N}_i(\xi,\eta)$$

Quadratic (serendipity) mapping

MUCH MORE ACCURATE THAN THE LINEAR MAP

NOTE A FEW THINGS:

- ON THE STRAIGHT EDGES, THE MAPPING REDUCES TO A LINEAR MAP
- IN THE INTERIOR OF AN ELEMENT, THE MAPPING IS NONLINEAR AND THE JACOBIAN WILL BE A FUNCTION OF POSITION.
- USING THE SHAPE FUNCTIONS ENSURES THAT FOR TWO NEIGHBORING ELEMENTS, THE COMMON (MAPPED) EDGE IS THE SAME AND DOES NOT HAVE SPURIOUS GAPS.
- WE CAN DEFINE MAPPINGS OF ANY ORDER p_{map}
- CAN BE USED FOR ELEMENTS WITH ANY NUMBER OF CURVED EDGES

$$p_{map} = 1$$ ← Linear map

$$p_{map} < p$$ ← Sub-parametric map

$$p_{map} = p$$ ← Iso-parametric map

$$p_{map} > p$$ ← Super-parametric map

Commonly used mapping

NOTE THAT HERE SHAPE FUNCTIONS DEFINED IN THE MASTER ELEMENT ARE USED. WHAT HAPPENS TO COMPLETENESS IN PHYSICAL DOMAIN?

- THE 8-NODED MAPPING IS SENSITIVE TO ELEMENT DISTORTION
- THE 9-NODED MAPPING IS LESS SENSITIVE TO ELEMENT DISTORTION
- ISOPARAMETRIC FORMULATION IS ESSENTIAL FOR NONLINEAR PROBLEMS WHEN THE UPDATED SOLUTION PROCEDURE IS ADOPTED.
- THERE ARE OTHER FAMILIES OF MAPPINGS. E.g. TRANSFINITE BLENDING FUNCTION APPROACH

CONVERGENCE ISSUES:

NICE DOMAIN, SMOOTH EXACT SOLUTIONS

EXACT ERROR DEFINED AS $e = u - u_{FE}$

$$U(e) = U(u) - U(u_{FE}) = 2B(e,e)$$

ENERGY OF ERROR

$$\| e \| = \sqrt{B(e,e)} \Rightarrow \text{ENERGY NORM}$$

BASIC A-PRIORI ERROR ESTIMATE

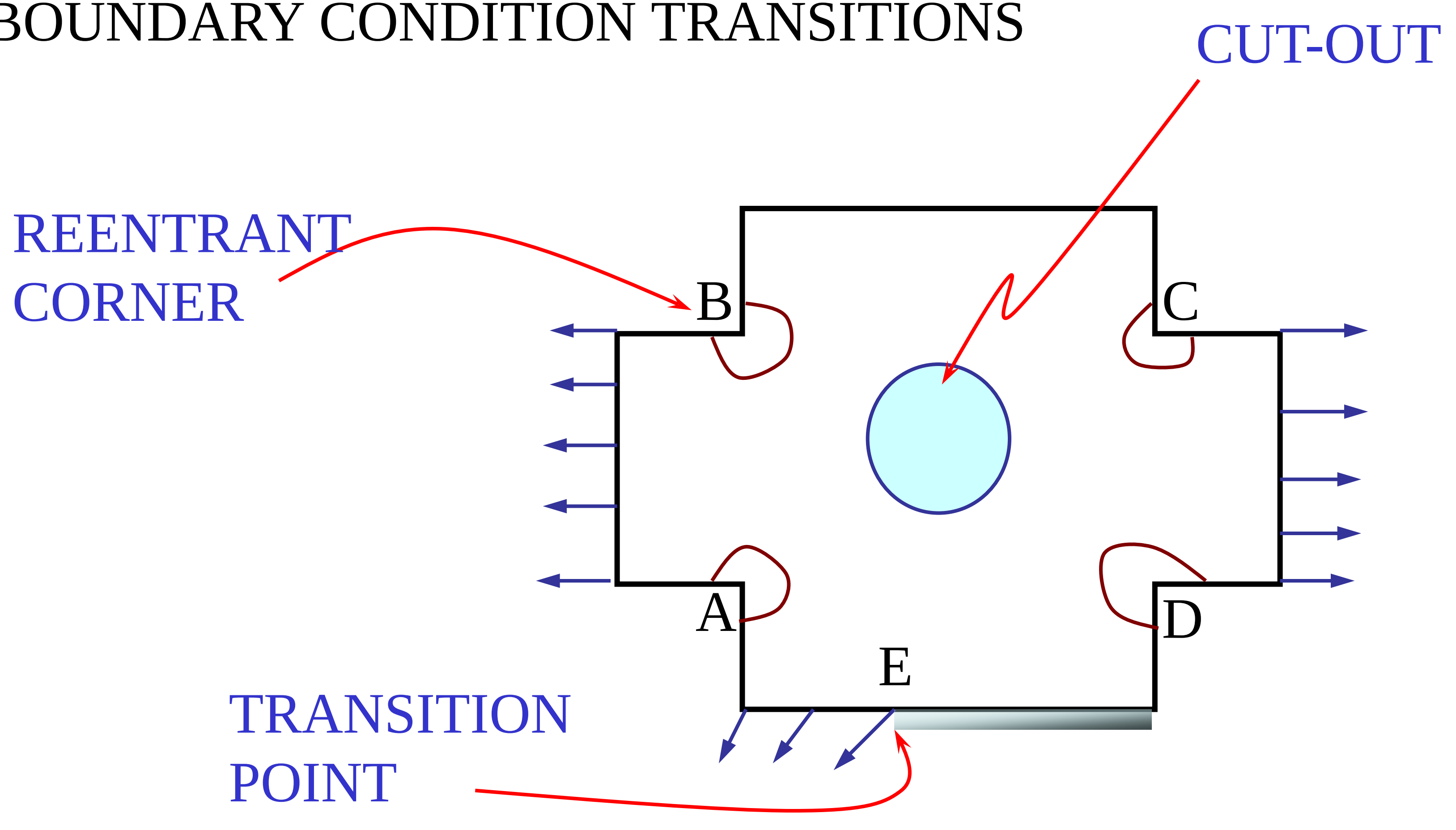
$$\| e \| \leq C h^{p}$$

APPROXIMATION ORDER (p)

MESH-SIZE (h)

SOURCES OF UNSMOOTHNESS OF SOLUTION:

- UNSMOOTH (OSCILLATING) MATERIAL DATA
- GEOMETRICAL DETAILS
- BOUNDARY CONDITION TRANSITIONS



$$\| e \| \leq C h^{\gamma}$$

$$\gamma = \min(p, \lambda)$$

POINTS A,B,C,D AND E ARE POINTS OF SINGULARITY

IN THE VICINITY OF EACH OF THESE POINTS, THE SOLUTION HAS THE FORM

$$u(r, \theta) = \sum_{i=1}^{\infty} K_i r^{\lambda_i} f(\lambda_i \theta)$$

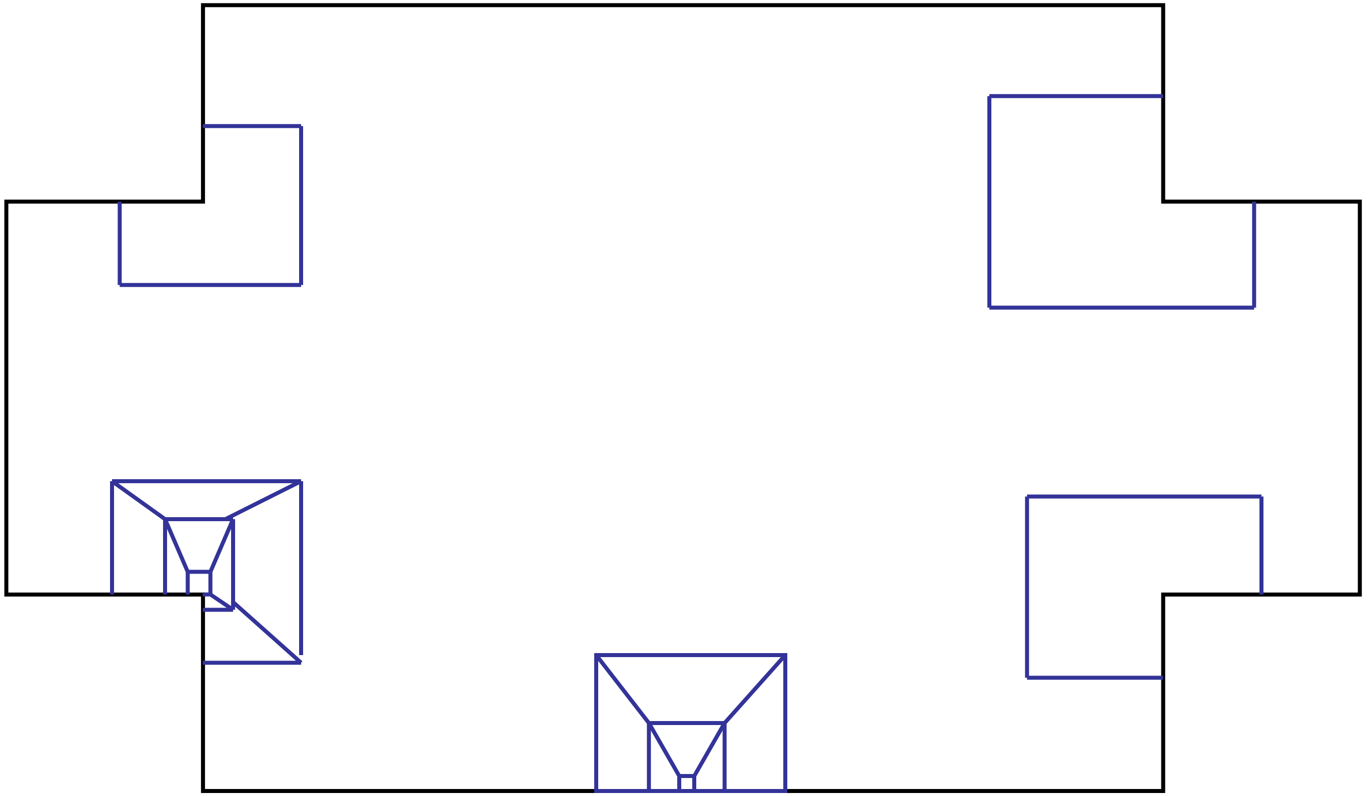
IN THE LOCAL POLAR COORDINATE SYSTEM

$\lambda_1 < 1$ FOR THESE VERTICES

THE CONVERGENCE RATE IS GREATLY DIMINISHED

UNIFORM MESH REFINEMENT IS NOT THE ANSWER

USE GEOMETRICALLY GRADED MESHES NEAR SINGULAR POINTS



PHYSICALLY THIS MEANS THAT IN THE REGIONS WITH HIGH DEFORMATION GRADIENTS, IT IS NECESSARY TO PUT A FINER MESH – A MESH THAT IS GEMOETRICALLY GRADED TOWARDS THE POINT OF SINGULARITY

WITH PROPER MESH DESIGN, THE CONVERGENCE CAN BE IMPROVED SIGNIFICANTLY

THIS MEANS MORE ACCURATE SOLUTIONS WITH SMALLER SIZED PROBLEMS

ESPECIALLY CRUCIAL FOR DOMAINS WITH MANY FEATURES AND STRUCTURAL DETAILS

hp-finite element methods