SHOW OFF TIME FOR GALERKIN APPROACH!

So, we have a weak formulation, i.e. $B(u,v) = F(v)$ with $u|_{\Gamma_D} = u_0 = 0$ (Γ_D is the Dirichlet boundary)

Thus, $B(u-u_n, u-u_n) = B(u,u) + B(u_n, u_n) - 2B(u, u_n)$

Since orthogonality of error gives $B(u-u_n, u_n) = 0$ (with u_n as a SPECIAL choice of v_n).

Thus $\boxed{B(u, u_n) = B(u_n, u_n)}$ or $B(\underbrace{u-u_n}_{e_n}, \underbrace{u-u_n}_{e_n}) = \|e_n\|^2_\Omega$

$B(u,u) - B(u_n, u_n) \geq 0$

$\Rightarrow B(u,u) \geq B(u_n, u_n)$.

Now let $\bar{u}_n$ be "0" solution obtained from "some" method such that $\bar{u}_n(x) = \sum_{i=0}^{N} \bar{\alpha}_i \cdot \varphi_i(x)$. Then $\tilde{e}_n = u - \bar{u}_n = u - u_n + u_n - \bar{u}_n$

$\Rightarrow \tilde{e}_n = e_n + e_n^*$ $\quad [e_n = u - u_n ;\ e_n^* = u_n - \bar{u}_n]$

Thus $B(\tilde{e}_n, \tilde{e}_n) = B(e_n + e_n^*, e_n + e_n^*) = B(e_n, e_n) + B(e_n^*, e_n^*) + 2B(e_n, e_n^*)$; $2B(e_n, e_n^*) = 0$ (by orthogonality and $v_n = e_n^*$)

$\Rightarrow \|\tilde{e}_n\|^2_\Omega = \|e_n\|^2_\Omega + \|e_n^*\|^2_\Omega \quad \Rightarrow \boxed{\|\tilde{e}_n\|_\Omega \geq \|e_n\|_\Omega}$ — (A)

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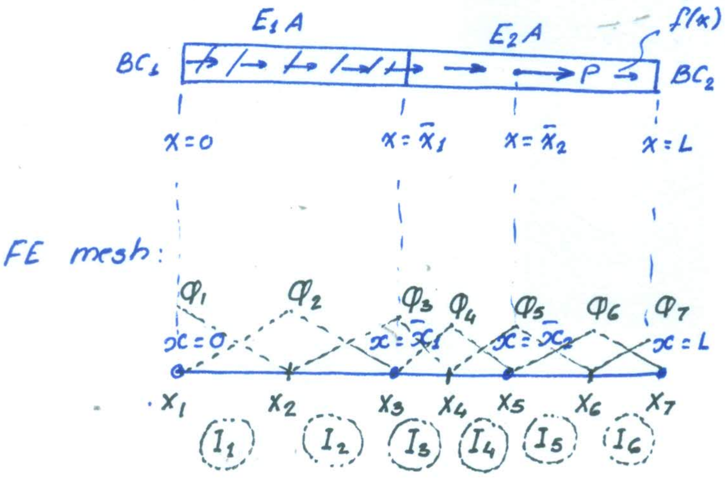
* Result (A) means that the Finite element solution is the <u>CLOSEST</u> to $u(x)$, of all the possible solutions of the form $\bar{u}_n(x)$, in ~~the~~ terms of the strain energy. The FE/Galerkin solution is said to be the BEST APPROXIMATION of $u(x)$

Let us now look at the FE solution of a specific bar problem:



BC_1 : $u(0) = 0$ (say)

BC_2 : $E_2 A \dfrac{du}{dx}\Big|_L = P_L$

{ A is constant }

* NOTE that points of MATERIAL transition and application of point loads HAVE to be nodes.

Let $u_h(x) = \sum_{i=1}^{7} \alpha_i \varphi_i(x)$ with $\alpha_1 = 0$ as $u(0) = 0$

Let $B(u_h, v_h) = F(v_h)$ with $B(u_h, v_h) = \int_0^{\bar{x}_1} E_1 A \dfrac{du_h}{dx}\dfrac{dv_h}{dx}\,dx + \int_{\bar{x}_1}^{\bar{x}_2} E_2 A \dfrac{du_h}{dx}\dfrac{dv_h}{dx}\,dx + \int_{\bar{x}_2}^{L} E_2 A \dfrac{du_h}{dx}\dfrac{dv_h}{dx}\,dx$; $F(v_h) = \int_0^L f\, v_h\,dx + P_L v_L + P v\big|_{\bar{x}_2}$ —(1)

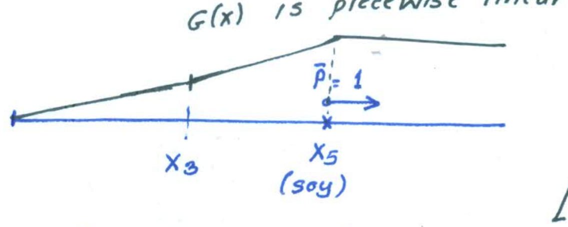
Let us also solve the problem for a different loading, given by $f(x) = 0$; $P_L = 0$; $P = 0$ and point load $\bar{P} = 1$ at $x = x_k$ (where k is a node of the mesh). Let the exact solution of this problem be $G(x)$ and its finite element solution $G_h(x)$ with:

$G(x)\big|_0 = G_h(x)\big|_0 = 0$; $G_h(x) = \sum_{i=1}^{7} \beta_i \varphi_i(x)$ [$\beta_1 = 0$ from $G(0) = 0$]

$\therefore$ G and G_h satisfy $B(G, v_h) = 1 \cdot v_h\big|_{x=x_k}$; $B(G_h, v_h) = 1 \cdot v_h\big|_{x_k}$ —(2)

$\Rightarrow B(G - G_h, v_h) = 0$. —(2a)

What is G ?



$G(x)$ is piecewise linear $\Rightarrow$ as $\bar{P}$ is at a node, $G_h(x) = G(x)$

[from completeness, best approximation]

$\Rightarrow B(u - u_h, v_h) = 0$ gives $B(u - u_h, G_h) = 0 = B(u - u_h, G)$.

$\Rightarrow B(G, u - u_h) = 1 \cdot (u - u_h)\big|_{x_k} = 0 \Rightarrow \boxed{u\big|_{x_k} = u_h\big|_{x_k}}$ NODAL EXACTNESS!