TIME DEPENDENT PROBLEMS

(A) **Parabolic** $\rightarrow$ $\boxed{C_1 \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(C_2 \frac{\partial u}{\partial x}\right) = r}$, $0 < x < L$

$$\underbrace{u(x,0) = \hat{u}(x)}_{\text{Initial Condition}} ; \quad \underbrace{u(0,t) = u_0 ; \quad C_2 \frac{\partial u}{\partial x}\Big|_L = P_L}_{\text{Boundary Conditions}}$$

$\boxed{\text{IBVP}}$

$u(x,t)$ is a function of position 'x' and time 't'

~ Example : 1D heat conduction problem

{ Is there an ENERGY functional, of which $u(x,t)$ is a minimizer ? }

(B) HYPERBOLIC $\rightarrow$ $\boxed{C_3 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(C_2 \frac{\partial u}{\partial x}\right) = r}$, $0 < x < L$

$$\underbrace{u(x,0) = \hat{u}(x) ; \quad \frac{\partial u}{\partial t}(x,0) = v_0(x)}_{\text{Initial conditions}} ; \quad \underbrace{u(0,t) = u_0 ; \quad C_2 \frac{\partial u}{\partial x}\Big|_L = P_L}_{\text{Boundary conditions}}$$

$\Downarrow$

How to solve these ? THE SAME WAY AS BEFORE!

Instantaneous weighted residual form :

(A) $$\int_0^L \left(C_1 \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(C_2 \frac{\partial u}{\partial x}\right)\right) v \, dx = \int_0^L r v \, dx$$

$u(x,t) \rightarrow$ function of both x and t

$\Rightarrow$ Spatial term $\rightarrow$ Integrate by parts.

$$\int_0^L (c_1 \dot{u}\, v + c_2 u' v')\, dx = \int_0^L r v\, dx + P_L\, v\big|_L$$

$\left\{ \dot{u} = \frac{\partial u}{\partial t} \;;\; u' = \frac{\partial u}{\partial x} \right\}$ with $u(x,t)$ satisfying the initial and Dirichlet conditions

$$u_{FE}(x,t) = \sum_{i=1}^{N} d_i(t)\, \varphi_i(x)$$

coefficient is function of time

$$v_{FE}(x,t) = \sum_{i=1}^{N} \beta_i\, \varphi_i(x)$$

not a function of time, β_i chosen by us

$$\Rightarrow \int_0^L \sum_{j=1}^{N} \left\{ c_1 (\dot{d}_j\, \varphi_j \varphi_i) + c_2\, d_j\, \varphi_j' \varphi_i' \right\} dx = \int_0^L r \varphi_j\, dx + P_L \cdot \varphi_j\big|_L$$

for $v = \varphi_i(x)$ as a specific choice

$\sim i^{th}$ equation OR

$$\sum_{j=1}^{N} \int_0^L \Big\{ \underbrace{(c_1 \varphi_j \varphi_i)}_{M_{ij}^{(1)}} \dot{d}_j + \underbrace{(c_2 \varphi_j' \varphi_i')}_{K_{ij}} d_j \Big\} dx = \underbrace{\int_0^L r \varphi_j\, dx + P_L\, \delta_{jN}}_{F_i}$$

$$\Rightarrow \boxed{[M^{(1)}]\{\dot{d}\} + [K]\{d\} = \{F\}}$$

with $u_{FE}(x,0) = \hat{u}(x)$; $u_{FE}(0,t) = u_0(t)$

~ SEMI - DISCRETE FORM $\Rightarrow$ SYSTEM OF ODE's

to be solved for

Element level calculations: (Element I_e)

$$[m^{(e)}] \rightarrow m_{ij}^{(e)} = \int_{x_e}^{x_{e+1}} C_1 N_i^{(e)} N_j^{(e)} \, dx$$

$$[k^{(e)}] \rightarrow k_{ij}^{(e)} = \int_{x_e}^{x_{e+1}} C_2 N_i^{(e)\prime} N_j^{(e)\prime} \, dx$$

$$[f^{(e)}] \rightarrow f_i^{(e)} = \int_{x_e}^{x_{e+1}} r \, N_i^{(e)} \, dx$$

$$\Downarrow$$

Assemble as before to get the global matrices

SOLVE using <u>time - marching</u>, or given $\{d(t)\}$

$\{d(t+\Delta t)\}$ is obtained by solving the ODE's at these time stations.

$*$ START from $\{d(0)\}$ $\longleftarrow$ known from I.C.

Here $d_i(0) = \hat{u}(x_i)$, but for general case use nodal interpolation with projection elsewhere