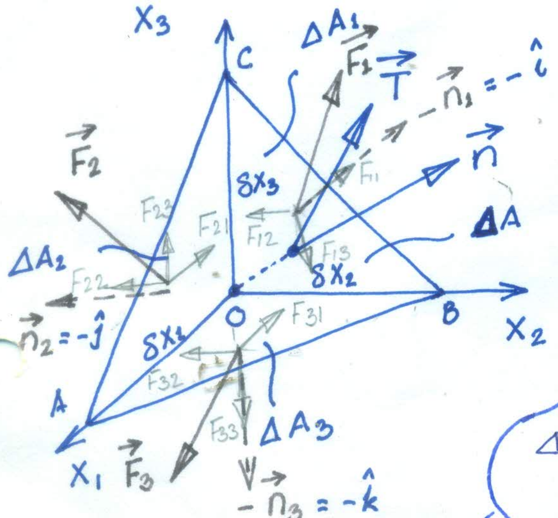
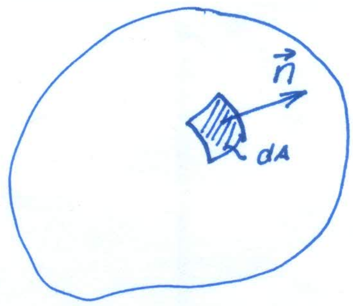
ALL ABOUT STRESS!





$$\Delta A_1 = \tfrac{1}{2}\,\delta x_2 \cdot \delta x_3 ; \quad \Delta A_2 = \tfrac{1}{2}\,\delta x_1 \cdot \delta x_3 ;$$

$$\Delta A_3 = \tfrac{1}{2}\,\delta x_1 \cdot \delta x_2$$

$$\Delta A \vec{n} = \vec{\Delta A} = \tfrac{1}{2}(\vec{AB} \times \vec{AC})$$

$$= \tfrac{1}{2}\left(-\delta x_1 \hat{i} + \delta x_2 \hat{j}\right) \times \left(-\delta x_1 \hat{i} + \delta x_3 \hat{k}\right)$$

$$= \Delta A_1 \hat{i} + \Delta A_2 \hat{j} + \Delta A_3 \hat{k}$$

Now $\Delta A_1 = \Delta A \cdot n_1$; $\Delta A_2 = \Delta A \cdot n_2$; $\Delta A_3 = \Delta A \cdot n_3$

where $\vec{n} = n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}$

$\therefore$ force balance:

$$\underset{\delta x_i \to 0}{Lt} \quad \vec{T}\,\cancel{\Delta A} + \vec{F_1}\,\underbrace{\Delta A_1}_{\cancel{\Delta A}\,n_1} + \vec{F_2}\,\underbrace{\Delta A_2}_{\cancel{\Delta A}\,n_2} + \vec{F_3}\,\underbrace{\Delta A_3}_{\cancel{\Delta A}\,n_3} + \frac{\Delta V}{\Delta A} \cdot \vec{f} = 0$$

$$\Rightarrow \vec{T} = -\left(\vec{F_1}\,n_1 + \vec{F_2}\,n_2 + \vec{F_3}\,n_3\right)$$

$$-F_{11}\hat{i} - F_{12}\hat{j} - F_{13}\hat{k} \qquad -F_{21}\hat{i} - F_{22}\hat{j} - F_{23}\hat{k} \qquad -F_{31}\hat{i} - F_{32}\hat{j} - F_{33}\hat{k}$$

$$= \left(F_{11} n_1 + F_{21} n_2 + F_{31} n_3\right)\hat{i} + \left(F_{12} n_1 + F_{22} n_2 + F_{32} n_3\right)\hat{j}$$

$$+ \left(F_{13} n_1 + F_{23} n_2 + F_{33} n_3\right)\hat{k}$$

$$= T_1 \hat{i} + T_2 \hat{j} + T_3 \hat{k}$$

$$\therefore \boxed{T_i = F_{ji}\, n_j}$$

F_{ji}: stress (force per unit area)

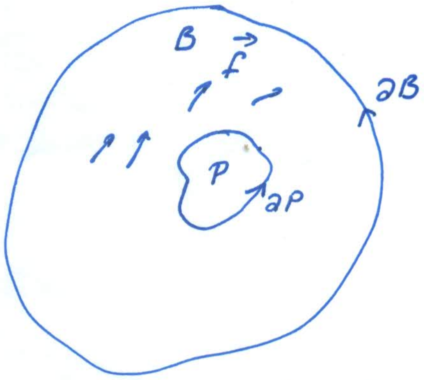
$$= \sigma_{ji}\, n_j = [\sigma]^T \{n\}$$

σ_{ji}: Cauchy Stress

Cauchy's hypothesis: The local, pointwise traction vector $\vec{T}$ is a linear function of the normal vector $\vec{n}$. $[\sigma]$ is the operator and is called the stress tensor.

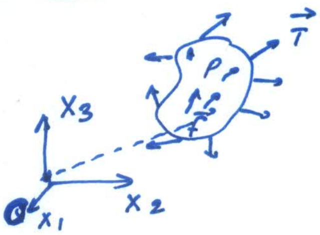
This relationship is true for any PORTION P of B, i.e.

$$\vec{T}\Big|_{\partial P} = [\sigma]^T \vec{n}\Big|_{\partial P}$$



Now the portion also has to be in equilibrium or

$$\int_P \vec{f}\, dV + \int_{\partial P} \vec{T}\, dA = \vec{0}$$



$$\Rightarrow \int_P \vec{f}\, dV + \int_{\partial P} [\sigma]^T \vec{n}\, dA = \vec{0}$$

$$\Downarrow \quad \int_P f_i\, dV + \int_{\partial P} \sigma_{ji}\, n_j\, dA = 0 \quad ; i = 1,2,3$$

$$\Rightarrow \int_P (f_i + \sigma_{ji,j})\, dV = 0$$

$\leftarrow$ true for all arbitrary portions $\Rightarrow \boxed{f_i + \sigma_{ji,j} = 0}$

Force balance equation

Moment balance:

Take moment about origin

$$\Rightarrow \int_P \underbrace{\vec{r} \times \vec{f}}_{r_i f_j \vec{e}_k}\, dV + \int_{\partial P} \underbrace{\vec{r} \times \vec{T}}_{r_i T_j \vec{e}_k}\, dA = \vec{0}$$

(i, j, k in cyclic symmetry)

Let us look at a specific case:

$$\int_P r_1 f_2 \vec{e}_3 \, dV + \int_{\partial P} r_1 \underset{(\sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3)}{T_2} \vec{e}_3 \, dA$$

$$\downarrow$$

$$\int_P (\overset{X_1}{r_1} \sigma_{12})_{,1} + (\overset{X_1}{r_1} \sigma_{22})_{,2} + (\overset{X_1}{r_1} \sigma_{32})_{,3}$$

$$\int_P \left(\sigma_{12} + X_1 \underline{\sigma_{12,1}} + X_1 \underline{\sigma_{22,2}} + X_1 \underline{\sigma_{32,3}} \right) dV$$

$$\boxed{\int_P \left(\sigma_{12} + \cancel{X_1 (-f_2)} \right) dV} + \cancel{\int_P X_1 f_2 \, dV} = \int_P \sigma_{12} \, dV$$

Similarly, $\int_P - r_2 f_1 \vec{e}_3 \, dV + \int_{\partial P} - \overset{X_2}{r_2} \underset{(\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3)}{T_1} \vec{e}_3 \, dA$

$$= \int_P \cancel{- X_2 f_1 \vec{e}_3} \qquad - \int_P \left\{ (\sigma_{11} X_2)_{,1} + (\sigma_{21} X_2)_{,2} + (\sigma_{31} X_2)_{,3} \right\} dV$$

$$- \int_P X_2 \left(\cancel{\overset{-f_1}{\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3}}} \right) dV - \int_P \sigma_{21} \, dV = \int_P - \sigma_{21} \, dV$$

$\Rightarrow$ Corresponding to $\vec{e}_3$, we have $\int_P (\sigma_{12} - \sigma_{21}) \, dV = 0$

$\Rightarrow$ $\boxed{\sigma_{12} = \sigma_{21}}$ $\leftarrow$ Stress symmetry!

OR $\boxed{\sigma_{ij} = \sigma_{ji}}$ $\Rightarrow [\sigma]^T = [\sigma]$

This means that stress tensor is symmetric $\Rightarrow$
when we look for those specific directions $\vec{n}$ such that

$\boxed{\vec{T} = [\sigma] \vec{n} = \lambda \vec{n}}$ ~ i.e. $\vec{T}$ directed along $\vec{n}$

$\Rightarrow$ $\lambda_i \rightarrow$ EIGENVALUE , $\vec{n}^{(i)} \rightarrow$ EIGENVECTOR

$\underbrace{\text{3 perpendicular directions}}$ ~ PRINCIPAL DIRECTIONS