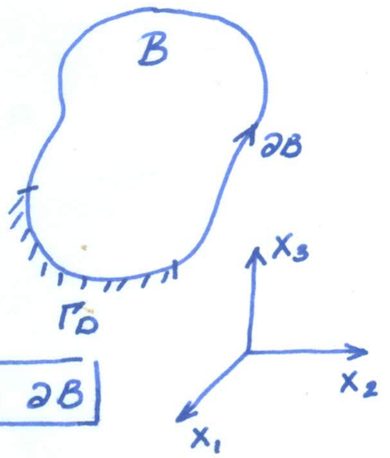
BOUNDARY VALUE PROBLEM

Conservation of momentum:



$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0 \quad \text{for } i = 1,2,3$$

$\langle$ 3 EQUATIONS $\rangle$

$\vec{u} = \vec{u}_0$ on Γ_D ← Displacement B.C.

$\sigma_{ji} n_j = T_i$ on Γ_N with $\boxed{\Gamma_N \cup \Gamma_D = \partial B}$

← Force B.C.

UNKNOWNS : σ_{ij} ← 6 unknowns ; u_i ← 3 unknowns

CONSTITUTIVE RELATIONSHIP:

$$\sigma_{ij} = C_{ijkl} \underbrace{\varepsilon_{kl}}_{\text{New unknowns}} \leftarrow 6 \text{ EQUATIONS}$$

New unknowns ← 6 unknowns

KINEMATICS:

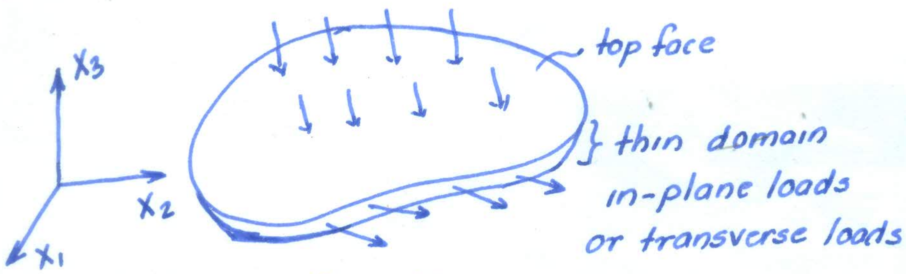
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \leftarrow 6 \text{ EQUATIONS}$$

$\Rightarrow$ UNKNOWNS $= 6 + 3 + 6 = 15$
EQUATIONS $= 3 + 6 + 6 = 15$ } solvable !!

Finally, we can solve for the displacement vector $\vec{u}(\vec{x})$ and recover everything else.

2^{nd} order PDE in $\vec{u}$ – 3 coupled equations

Not so trivial to solve for general domains. Can we IDEALIZE to reduce our effort?



PLANE STRESS

$\sigma_{3j} \approx 0$ as compared to other σ_{ij}

$\Rightarrow \sigma_{33} = 0$ (not ε_{33}, why?)

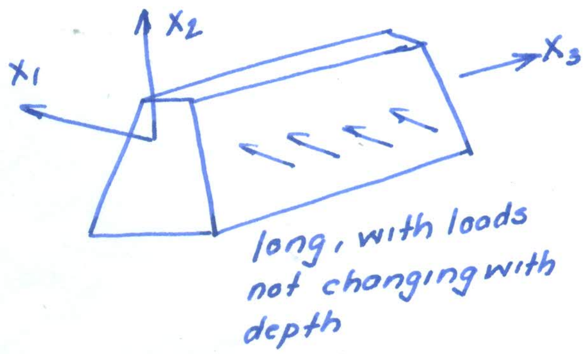
$\sigma_{31} = \sigma_{32} = 0$

$\Rightarrow \gamma_{31} = \gamma_{32} = 0 \quad (\gamma_{ij} = 2\varepsilon_{ij})$

$$\varepsilon_{33} = \frac{-\nu}{E}(\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22})$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11})$$



PLANE STRAIN

$\varepsilon_{3j} \approx 0$ as compared to other directions.

$\varepsilon_{33} = 0$ (not σ_{33}!)

$\varepsilon_{31} = \varepsilon_{32} = 0$

$\Rightarrow \sigma_{31} = \sigma_{32} = 0$

$$\varepsilon_{33} = \underbrace{\frac{1}{E}(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22}))}_{=0}$$

$$\Rightarrow \boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}$$

$$\Rightarrow \varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu(\nu\sigma_{11} + \nu\sigma_{22}))$$

$$= \frac{1}{E}((1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22})$$

$$= \boxed{\frac{(1-\nu^2)}{E}\left(\sigma_{11} - \frac{\nu}{(1-\nu)}\sigma_{22}\right)}$$

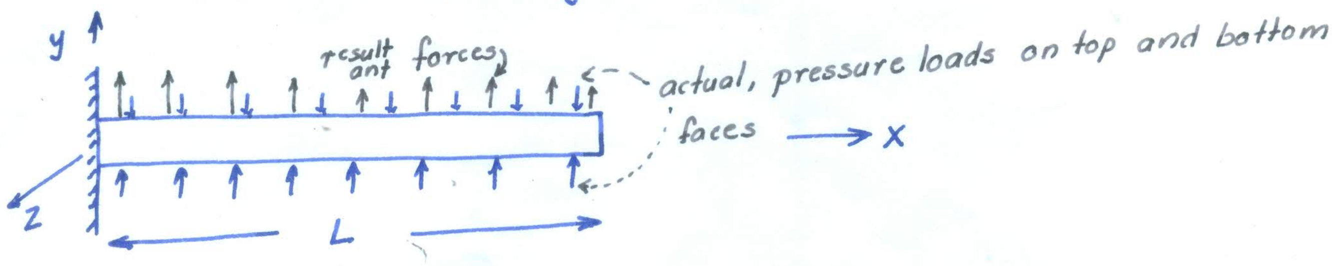
$$\Rightarrow \varepsilon_{22} = \frac{(1-\nu^2)}{E}\left(\sigma_{22} - \frac{\nu}{(1-\nu)}\sigma_{11}\right)$$

* For both cases find

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}}_{\text{ENGINEERING STRESS "VECTOR"}} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix} \underbrace{\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}}_{\text{ENGINEERING STRAIN "VECTOR"}}$$

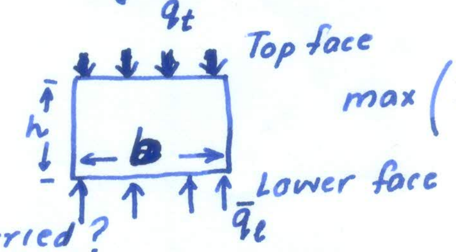
(Only a convenient form of writing!)

Let us now talk of long-slender members subjected to the transverse loads given:



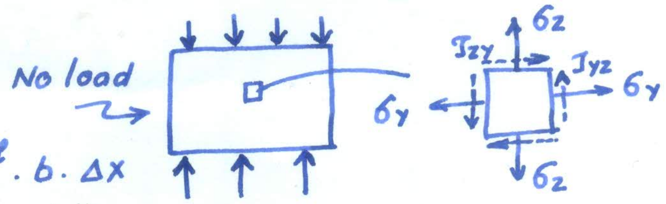
→ Pressure (and friction + drag) forces on the outer surface

→ CROSS-SECTION



$$\max\left(\frac{b}{L}, \frac{h}{L}\right) << 1$$

→ How will the load be carried?



$$\bar{q}_t \cdot b \cdot \Delta x \approx \Delta F_z^t$$

$$\bar{q}_\ell \cdot b \cdot \Delta x \approx \Delta F_y^\ell \approx \sigma_z^\ell \cdot b \cdot \Delta x$$

$$\Rightarrow \sigma_z^\ell \approx \bar{q}_\ell \; ; \; \sigma_z^t \approx \bar{q}_t$$