Consequences of strain derivation

Let $\overrightarrow{PQ_i} = \overrightarrow{\Delta X}^{(i)}$ and $\overrightarrow{P'Q_i'} = \overrightarrow{\Delta x}^{(i)}$, then

with $l_0^{(i)} = |\overrightarrow{\Delta X}^{(i)}|$; $l^{(i)} = |\overrightarrow{\Delta x}^{(i)}|$, we have

$$\frac{\Delta l^{(i)}}{l_0^{(i)}} = \frac{l^{(i)} - l_0^{(i)}}{l_0^{(i)}} = \frac{\overrightarrow{\Delta X}^{(i)\,T}\,[\varepsilon]\,\overrightarrow{\Delta X}^{(i)}}{l_0^{(i)\,2}}$$

where $[\varepsilon]\big|_P = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}_{\text{at } P}$ as the strain tensor at $\underline{\underline{P}}$.

- Knowing $[\varepsilon]\big|_P$ we can get stretch of ANY infinitesimal line element $\overrightarrow{PQ_i}$ at P.

E.g. $\overrightarrow{\Delta X}^{(1)} = \Delta X\,\hat{i} \Rightarrow \dfrac{\Delta l^{(1)}}{l_0^{(1)}} = \dfrac{\{\Delta X\ \ 0\ \ 0\}\begin{Bmatrix}\varepsilon_{11}\Delta X \\ \varepsilon_{21}\Delta X \\ \varepsilon_{31}\Delta X\end{Bmatrix}}{\Delta X^2}$

$$= \frac{\varepsilon_{11}\cdot\Delta X^2}{\Delta X^2} = \varepsilon_{11}$$

$\overrightarrow{\Delta X}^{(2)} = \dfrac{\Delta X}{\sqrt{2}}(\hat{i}+\hat{j}) \Rightarrow \dfrac{\Delta l^{(2)}}{l_0^{(2)}} = \dfrac{\Delta X^2\,\{1\ \ 1\ \ 0\}\begin{Bmatrix}\varepsilon_{11}+\varepsilon_{12} \\ \varepsilon_{21}+\varepsilon_{22} \\ \varepsilon_{31}+\varepsilon_{22}\end{Bmatrix}\times\frac{1}{2}}{\Delta X^2}$

$$= \boxed{\varepsilon_{11} + 2\varepsilon_{12} + \varepsilon_{22}}\,/2$$

and so on !!

Eigenvalue problem for strain tensor:

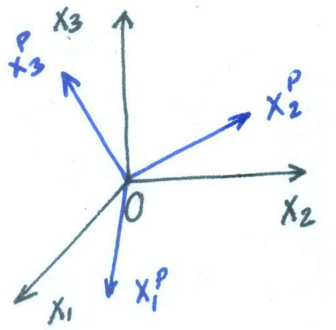
$$[\varepsilon]\vec{n} = \lambda \vec{n} \Rightarrow \text{we get eigenvalues } \lambda_1, \lambda_2, \lambda_3$$

and corresponding eigenvectors $\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)}$.

- Since $[\varepsilon]$ is symmetric, $\{\lambda_i\}_{i=1}^{3}$ is real and

$\vec{n}^{(1)} \perp \vec{n}^{(2)} \perp \vec{n}^{(3)} \Leftarrow$ form an orthogonal coordinate system

~~prin~~ principal coordinates / axes. $\longrightarrow$ $X_1^p - X_2^p - X_3^p$



Now $[R] = [\vec{n}^{(1)} \;\; \vec{n}^{(2)} \;\; \vec{n}^{(3)}]$

$$\Rightarrow [\varepsilon][R] = [\lambda_1\vec{n}^{(1)} \;\; \lambda_2\vec{n}^{(2)} \;\; \lambda_3\vec{n}^{(3)}]$$

$$= [n^{(1)} \;\; n^{(2)} \;\; n^{(3)}]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$$

$$\Rightarrow [\varepsilon][R] = [R][\Lambda] \Rightarrow [\varepsilon] = [R][\Lambda][R]^T$$

* Note that $\{\Delta x^p\} = [R]^T\{\Delta x\} = [Q]\{\Delta x\}$ with

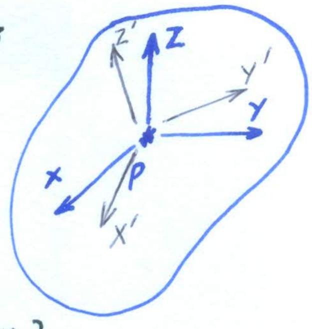
$$[Q] = [R]^T \Rightarrow [\varepsilon] = [Q]^T[\Lambda][Q]$$

or $\boxed{[\Lambda] = [Q][\varepsilon][Q]^T}$

* Note that $\vec{\Delta x}' = \Delta x\, \vec{n}^{(i)}$ gives $\dfrac{\Delta l}{l_0} = \boxed{\lambda_i}$

* No shear in principal strain coordinate system.

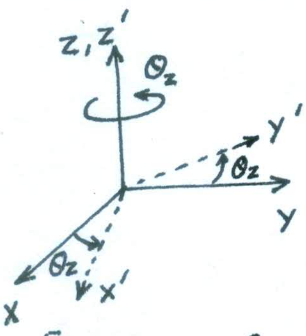
Stress and strain transformation



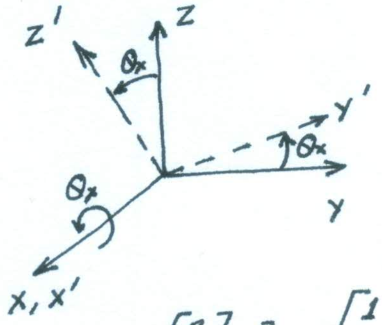
- Let point P have coordinates (X, Y, Z)
- Let (X', Y', Z') be coordinates in a new, rotated coordinate system obtained by:

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \underbrace{[R]}_{\text{rotation matrix}} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

Examples of $[R]$:



$$[R] = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$[\sigma'] = [R][\sigma][R]^T \quad ; \quad [\varepsilon'] = [R][\varepsilon][R]^T$$

Principal stress directions:

$$[\sigma]\{n\} = \lambda\{n\} \Rightarrow \text{Eigenvalues (principal stresses)} \quad \lambda_1, \lambda_2, \lambda_3$$

Principal directions (or eigenvectors) $\vec{n}^1, \vec{n}^2, \vec{n}^3$ with $|\vec{n}^1| = |\vec{n}^2| = |\vec{n}^3| = 1$

and $\vec{n}^1 \perp \vec{n}^2 \perp \vec{n}^3$.

$$\begin{Bmatrix} \vec{n}^1 \\ \vec{n}^2 \\ \vec{n}^3 \end{Bmatrix} = \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \\ n_1^3 & n_2^3 & n_3^3 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \Rightarrow \{\vec{n}\} = \underbrace{[N]}_{[R^P]}\{\hat{i}\}$$

$$\Rightarrow \{X^P\} = [R^P]\{X\}$$