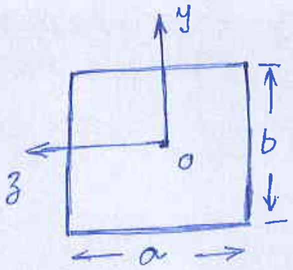
AN EXAMPLE OF MULTI - MATERIAL

Let us take a cantilevered beam of rectangular cross-section $(a \times b)$ and subjected to end moment M_z $(M_y = 0)$. Let the initial construction be of Aluminium with $E \sim 70\,GPa$, $\rho \sim 2700\,kg/m^3$.



· Location of centroid: At centre of cross-section

Sectional properties: $I_{yy} = \frac{1}{12} b a^3$;

$I_{yz} = 0$; $I_{zz} = \frac{1}{12} a b^3$

Moment curvature relationship:

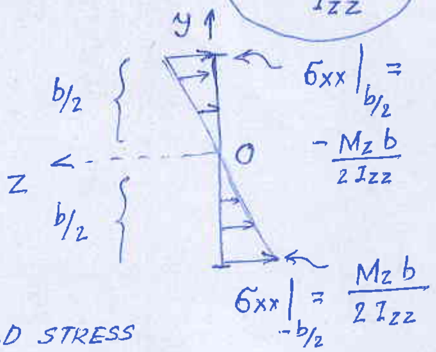
$$M_y = 0 = -E I_{yy} w'' \Rightarrow \boxed{w'' = 0}$$

$$M_z = E I_{zz} v'' \Rightarrow \boxed{v'' = \frac{M_z}{E I_{zz}}}$$

$$\Rightarrow \sigma_{xx} = E \varepsilon_{xx} = E(-y v'' - z w'') = -E v'' y = \left(-\frac{M_z y}{I_{zz}}\right)$$

Variation of σ_{xx}: Linear in y with maximum magnitude at $y = \pm b/2$ as

$$|\sigma_{xx}|_{max} = \frac{M_z b}{2 I_{zz}} = \boxed{\frac{6 M_z}{a b^2}}$$

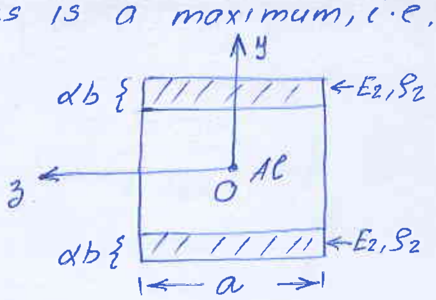


Let M_z be such that $|\sigma_{xx}|_{max} = \sigma_Y^{Al}$ ← YIELD STRESS OF Al.

$$\Rightarrow \boxed{M_z = \frac{\sigma_Y^{Al} \cdot (a b^2)}{6}}$$

* NOW WE WANT TO DO MATERIAL SUBSTITUTION TO MAKE $|\sigma_{xx}|_{max} < \sigma_Y$

⇒ Candidate location: Region where bending stress is a maximum, i.e. at top and bottom.



Candidate material: As the material strips act as springs in parallel, we would heuristically want to use a STIFFER material to reduce load share of the Al part.

Let $E_2 = \beta E_{Al}$; $\rho_2 = \gamma \rho_{Al}$ (with $\beta > 1$) as the choice

Sectional properties: Centroid: O (modulus weighted) is still at centre of section due to symmetry (work it out!).

$$E^* I^*_{yy} = \frac{1}{12} E\,(1-2\alpha)\,b\,a^3 + \left(\frac{1}{12} E_2\,(\alpha b)\,a^3\right) \times 2$$

$$= \frac{1}{12} E b a^3 \left[(1-2\alpha) + 2\alpha\beta\right] = E I_{yy}\,\Big(1 - \underbrace{2\alpha(1-\beta)}_{\delta_1}\Big)$$

$$E^* I^*_{yz} = 0$$

$$E^* I^*_{zz} = \frac{1}{12} E\,a\,\big((1-2\alpha)b\big)^3 + 2\left[\frac{1}{12} E_2\,a(\alpha b)^3 + E_2\,a(\alpha b)(1-\alpha)^2\left(\frac{b}{2}\right)^2\right]$$

$$= \frac{1}{12} E a b^3 \left[(1-2\alpha)^3 + 2\alpha^3\beta + 2\beta\alpha(1-\alpha)^2 \times 3\right]$$

$$= \frac{1}{12} E a b^3 \left[1 - 8\alpha^3(1-\beta) + 12\alpha^2(1-\beta) - 6\alpha(1-\beta)\right]$$

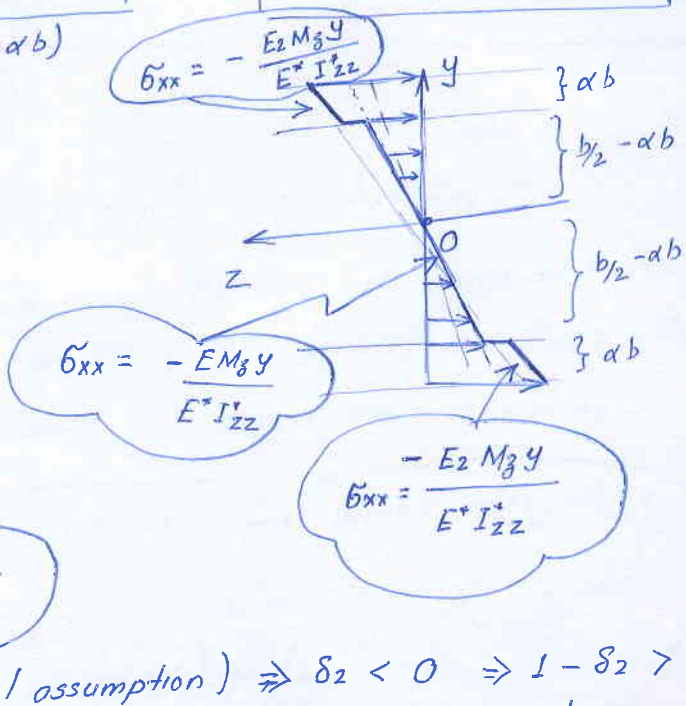
$$= \frac{1}{12} E a b^3 \left\{1 - (1-\beta)(8\alpha^3 - 12\alpha^2 + 6\alpha)\right\}$$

$$= E I_{zz}\,\Big(1 - \underbrace{(1-\beta)(8\alpha^3 - 12\alpha^2 + 6\alpha)}_{\delta_2}\Big)$$

Moment - curvature relationship: $M_y = 0 \Rightarrow w'' = 0$

$$M_z = E^* I^*_{zz}\, v'' \Rightarrow \boxed{v'' = \frac{M_z}{E^* I^*_{zz}}} \Rightarrow \boxed{\sigma_{xx} = -\tilde{E}(y)\cdot y\,\frac{M_z}{E^* I^*_{zz}}}$$

with $\tilde{E}(y) = \begin{cases} E_2 & \text{for } |y| > (b/2 - \alpha b) \\ E & \text{otherwise} \end{cases}$



* Notice JUMP in σ_{xx} across the material boundaries.

Look at stress:

In Al : $\sigma_{xx} = -\dfrac{E M_z y}{E^* I^*_{zz}}$

$$= -\frac{M_z y}{I_{zz}(1-\delta_2)}$$

As $\alpha \ll 1$ and $\beta > 1$ (initial assumption) $\Rightarrow \delta_2 < 0 \Rightarrow 1 - \delta_2 > 1$

$\Rightarrow \sigma_{xx}|_{Al}$ is LESSER than what was there earlier!

** Also, with $|y| \le b/2 - \alpha b$, now no point is CRITICALLY stressed in this region (SHOW THIS!!)

In the "new" material region, i.e. $|y| > (b/2 - \alpha b)$

$$\sigma_{xx} = -\frac{E_2 M_z y}{E^* I_{zz}^*} = -\frac{\beta}{(1-\delta_2)} \frac{M_z y}{I_{zz}}$$

Thus, the replaced material sees a stress MAGNIFICATION by a factor $\dfrac{\beta}{(1-\delta_2)}$

$$|\sigma_{xx}|_{MAX} = \frac{\beta}{(1-\delta_2)} \frac{M_z \cdot b/2}{I_{zz}} = \frac{\beta}{(1-\delta_2)} \cdot \sigma_Y^{Al} > \sigma_Y^{Al} \;!!$$

$$\boxed{\text{Thus, the replacement material should have } \sigma_Y > \frac{\beta}{(1-\delta_2)} \sigma_Y^{Al}}$$

Choice of material : Let replacement material be spring steel with

$E_2 = E_{ss} \simeq 210$ GPa ; $\sigma_Y^{ss} \sim$ ~~1800~~ 1200 MPa ; $\rho_{ss} \sim 8000$ kg/m^3

$(\sigma_Y^{Al} \sim 280 \text{ MPa}) \Rightarrow \beta \approx 3$

Choosing $\alpha = 0.05$, $-\delta_2 = 2(8 \times 0.05^3 - 12 \times 0.05^2 + 6 \times 0.05) \approx 0.54$

$$\therefore \frac{\beta}{(1-\delta_2)} \approx \frac{3}{1.54} \sim 2$$

Now $\sigma_Y^{ss} = 1200$ MPa $\approx 4.3\, \sigma_Y^{Al} \Rightarrow |\sigma_{xx}|_{max} < \sigma_Y^{ss}$.

* Since we have this margin, we can REDUCE α more (i.e. thinner strip of steel) in order to REDUCE weight penalty as $\rho_{ss} \approx 3\rho_{Al}$

— This is an example of design decision making

* WHY DON'T YOU FIND the α for which everywhere the $|\sigma_{xx}|_{max} < 0.6\ \sigma_Y$

(Factor of safety) $\sim \dfrac{1}{1.5}$