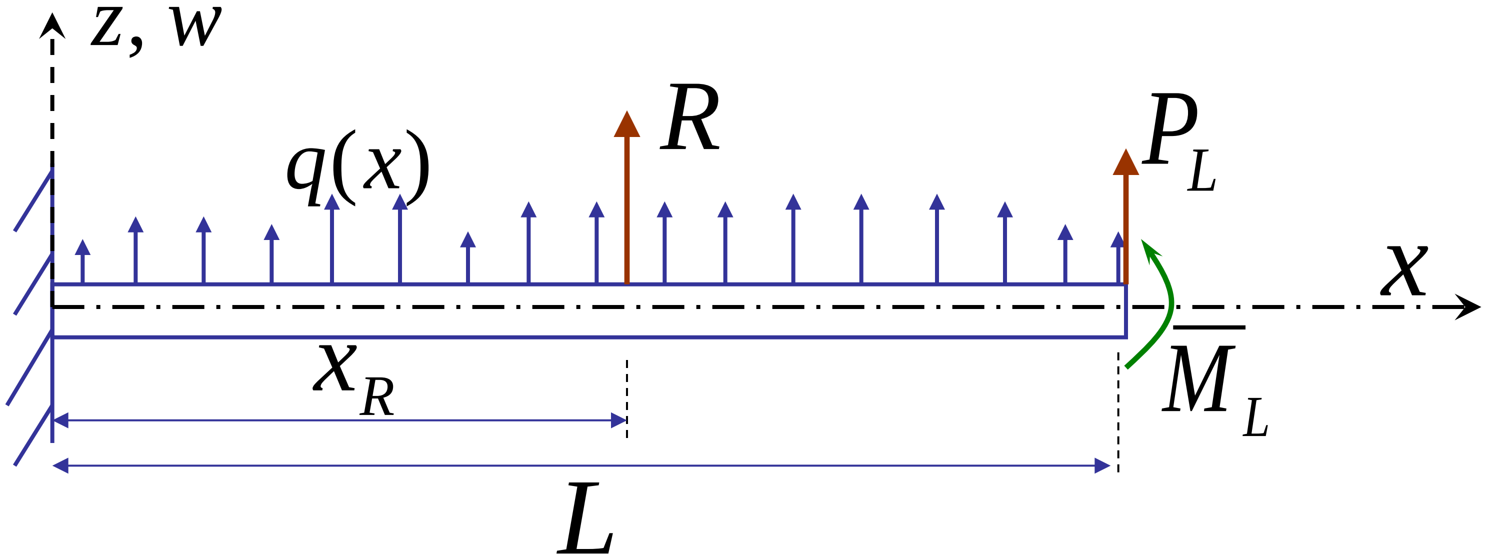
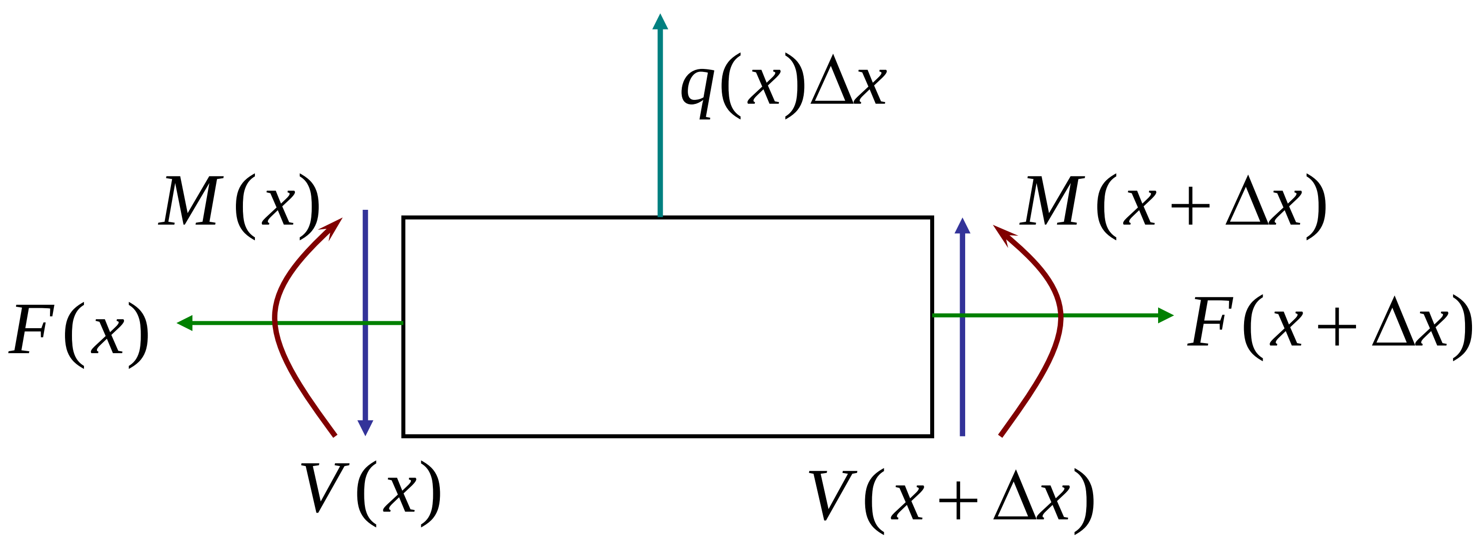
ONE-DIMENSIONAL FINITE ELEMENT ANALYSIS

- REVIEW LAST LECTURE
- A SIMPLE BEAM EXAMPLE
- VARIATIONAL OR ENERGY METHOD
- THE MEMBRANE PROBLEM
- WEAK FORMULATION



BEAM WITH TRANSVERSE LOADING



$$r(x) = \frac{d^2}{dx^2}\left(EI \frac{d^2 w}{dx^2} \right) - q(x) = 0$$

THE RESIDUAL

$$\int_{x=0}^{L} r(x)\chi(x)dx = 0$$

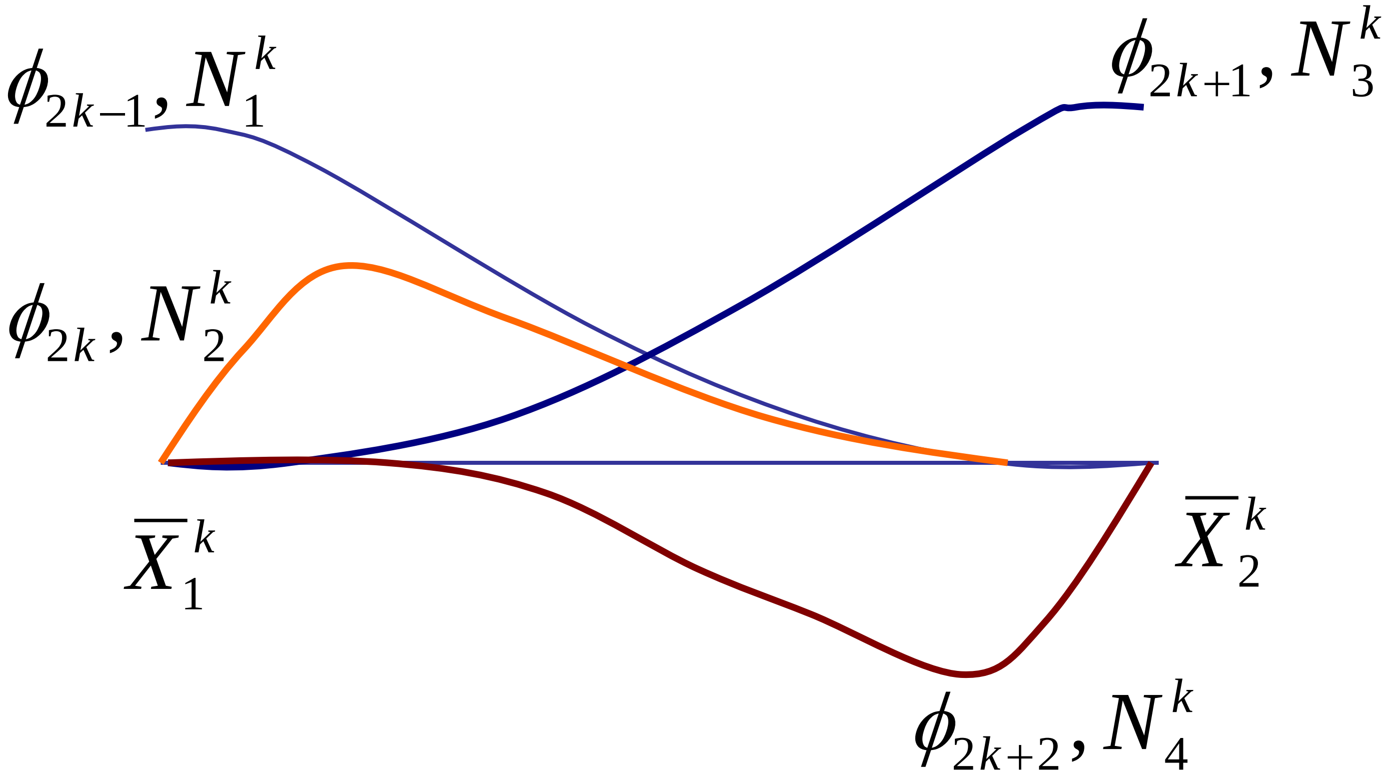
WEIGHTED RESIDUAL FORM

THE WEAK FORM (VIRTUAL WORK)

$$\int_{x=0}^{L} EI \frac{d^2 w}{dx^2}\frac{d^2 \chi}{dx^2} dx =$$

$$\int_{x=0}^{L} q\chi dx + \left(\underline{-\frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right)} \chi \right)\Bigg|_{x=0}^{x=L} + \left(\underline{EI \frac{d^2 w}{dx^2}} \frac{d\chi}{dx} \right)\Bigg|_{x=0}^{x=L}$$

V M



THE ODD SHAPE FUNCTIONS CORRESPOND TO THE VALUE AT THE NODE (derivative zero)

THE EVEN SHAPE FUNCTIONS CORRESPOND TO THE DERIVATIVE AT THE NODE (value zero)

THE HERMITE CUBIC SHAPE FUNCTIONS ARE:

$$N_1^k(x) = 1 - 3\frac{(x-\overline{X}_1^k)^2}{h_k^2} + 2\frac{(x-\overline{X}_1^k)^3}{h_k^3}$$

$$N_2^k(x) = (x-\overline{X}_1^k) - 2\frac{(x-\overline{X}_1^k)^2}{h_k} + \frac{(x-\overline{X}_1^k)^3}{h_k^2}$$

$$N_3^k(x) = 3\frac{(x-\overline{X}_1^k)^2}{h_k^2} - 2\frac{(x-\overline{X}_1^k)^3}{h_k^3}$$

$$N_4^k(x) = -\frac{(x-\overline{X}_1^k)^2}{h_k} + \frac{(x-\overline{X}_1^k)^3}{h_k^2}$$

GLOBAL

$$w_{FE}(x) = \sum_{i=1}^{2N} \alpha_i \phi_i(x)$$

ELEMENT

$$w_{FE}\,|_{I_k} = \sum_{i=1}^{4} \overline{\alpha}_i^{\,k} N_i^k(x)$$

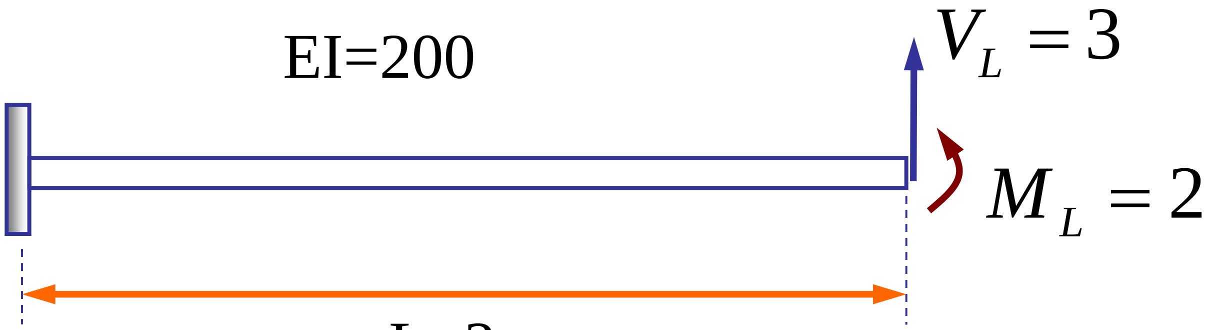
CONNECTIVITY INFORMATION

$$\overline{\alpha}_1^{\,k} \Leftrightarrow \alpha_{2k-1} \qquad \overline{\alpha}_2^{\,k} \Leftrightarrow \alpha_{2k} \qquad \overline{\alpha}_3^{\,k} \Leftrightarrow \alpha_{2k+1} \qquad \overline{\alpha}_4^{\,k} \Leftrightarrow \alpha_{2k+2}$$

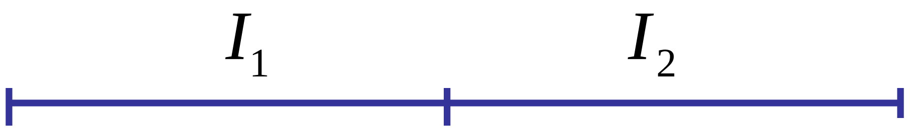
$$[K^{(k)}] = \frac{EI}{h_k^3} \begin{bmatrix} 12 & 6h_k & -12 & 6h_k \\ 6h_k & 4h_k^2 & -6h_k & 2h_k^2 \\ -12 & -6h_k & 12 & -6h_k \\ 6h_k & 2h_k^2 & -6h_k & 4h_k^2 \end{bmatrix}$$

Element Stiffness Matrix

A SIMPLE EXAMPLE



$$w = \frac{dw}{dx} = 0$$



$X_1 = 0$ $\quad X_2 = 0$ $\quad X_3 = 0$

Element-size

$$h_1 = h_2 = 1$$

$$w_{FE}(x) = \sum_{i=1}^{6} \alpha_i \phi_i(x)$$

$$w_{FE}\big|_{I_1}(x) = \sum_{i=1}^{4} \alpha_i \phi_i(x) = \sum_{i=1}^{4} \overline{\alpha}_i^1 N_i^1(x)$$

$$w_{FE}\big|_{I_2}(x) = \sum_{i=3}^{6} \alpha_i \phi_i(x) = \sum_{i=1}^{4} \overline{\alpha}_i^2 N_i^2(x)$$

$$
[K^{(1)}] = 200 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} = [K^{(2)}]
$$

$$
[K] = 200 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}
$$

$$\{F\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \end{Bmatrix}$$

SOLVE THE SYSTEM OF EQUATIONS NOW

WE DID THE PRINCIPAL OF VIRTUAL WORK, WHICH IS GENERAL IN NATURE

IN MANY CASES IT IS POSSIBLE TO WORK WITH AN ENERGY MINIMISATION PROCEDURE (LINEAR PROBLEMS OF MECHANICS)

MINIMISATION OF TOTAL POTENTIAL ENERGY

TOTAL POTENTIAL ENERGY $\pi = U + V$

$$U = \frac{1}{2}\int_B \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right) dV$$

$$V = -\int_B \left(b_x u_x + b_y u_y + b_z u_z\right) dV - \int_{\partial B} \left(T_x u_x + T_y u_y + T_z u_z\right) dA$$

U IS THE STRAIN ENERGY FOR THE BODY

V IS THE (NEGATIVE) OF THE WORK DONE BY THE EXTERNAL FORCES (BOTH BODY AND SURFACE FORCES)

SIMPLER CASES: 1D EXAMPLES

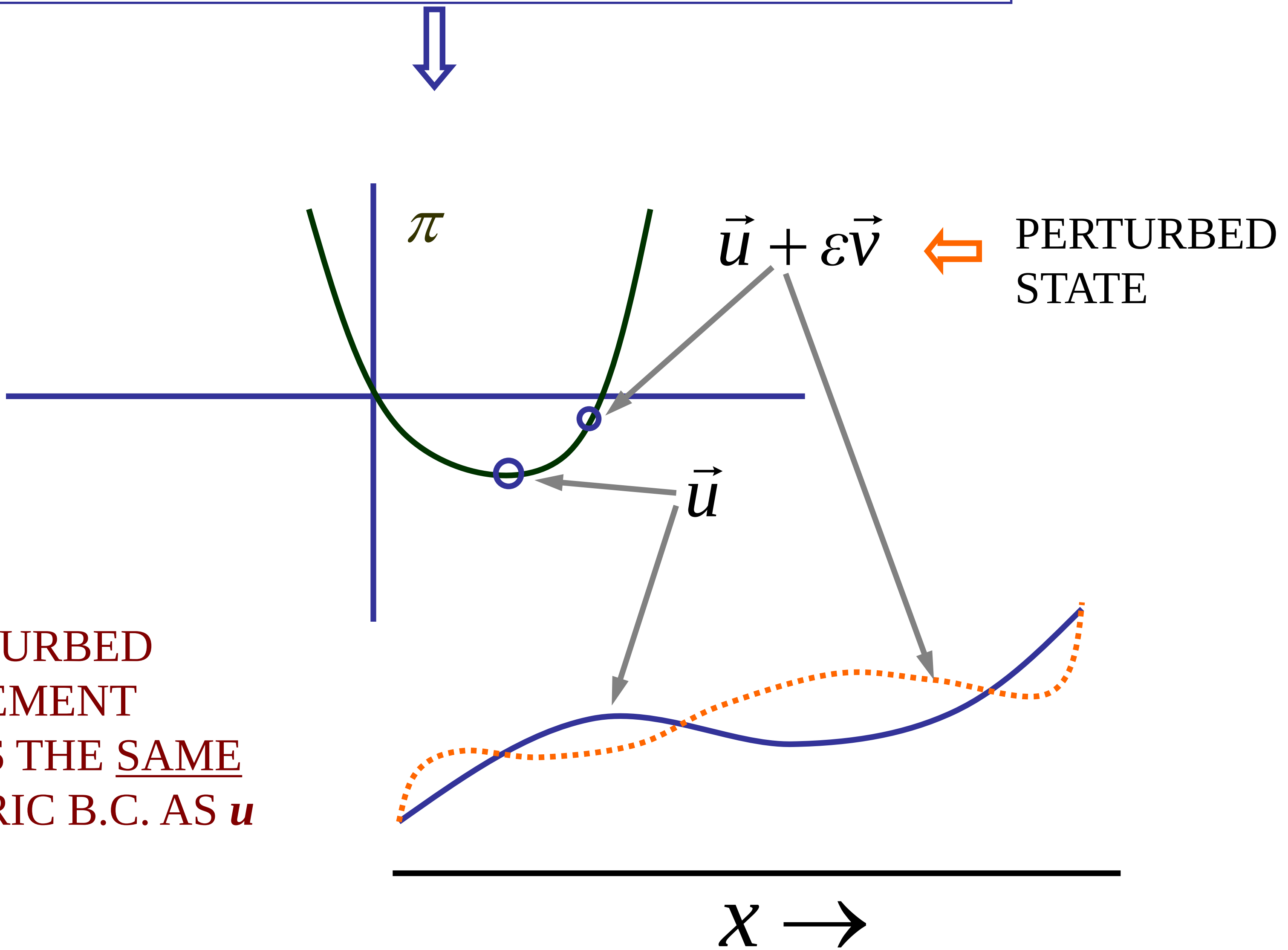
BAR PROBLEM

$$\pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L (fu)\,dx - P_L u(L)$$

$$\pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L (qw)\,dx - V_L w(L) - M_L \frac{dw}{dx}\Big|_{x=L}$$

BEAM PROBLEM

THE EXACT SOLUTION IS MINIMIZER OF π



THE PERTURBED DISPLACEMENT SATISFIES THE SAME GEOMETRIC B.C. AS ***u***

$$\pi(\vec{u}+\varepsilon\,\vec{v}) \approx \pi(\vec{u}) + \frac{\partial \pi}{\partial \varepsilon}\Big|_{\varepsilon=0}\,\varepsilon + \text{higher order terms in } \varepsilon$$

First variation of π

$$\delta^{(1)}\pi = \lim_{\varepsilon\to 0}\frac{\left[\pi(\vec{u}+\varepsilon\,\vec{v})-\pi(\vec{u})\right]}{\varepsilon} = 0$$

CONDITION FOR EQUILIBRIUM

EXAMPLE OF THE BAR PROBLEM

$$\pi(u+\varepsilon v) = \frac{1}{2}\int_0^L EA\left(\frac{d(u+\varepsilon v)}{dx}\right)^2 dx - \int_0^L (f\,(u+\varepsilon v))\,dx - P_L\,(u(L)+\varepsilon v(L))$$

FOR MINIMA, THE FIRST VARIATION HAS TO BE ZERO

$$\delta^{(1)}\pi = \int_0^L EA\frac{du}{dx}\frac{dv}{dx}dx - \int_0^L fv\,dx - P_L v_L = 0$$

$$u(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) \qquad v(x) = \delta u(x) = \sum_{i=1}^{N} \delta\alpha_i\, \phi_i(x)$$

Perturbation or change in displacement *u*

Perturbation or change in the coefficients

THE VARIATIONAL FORMULATION

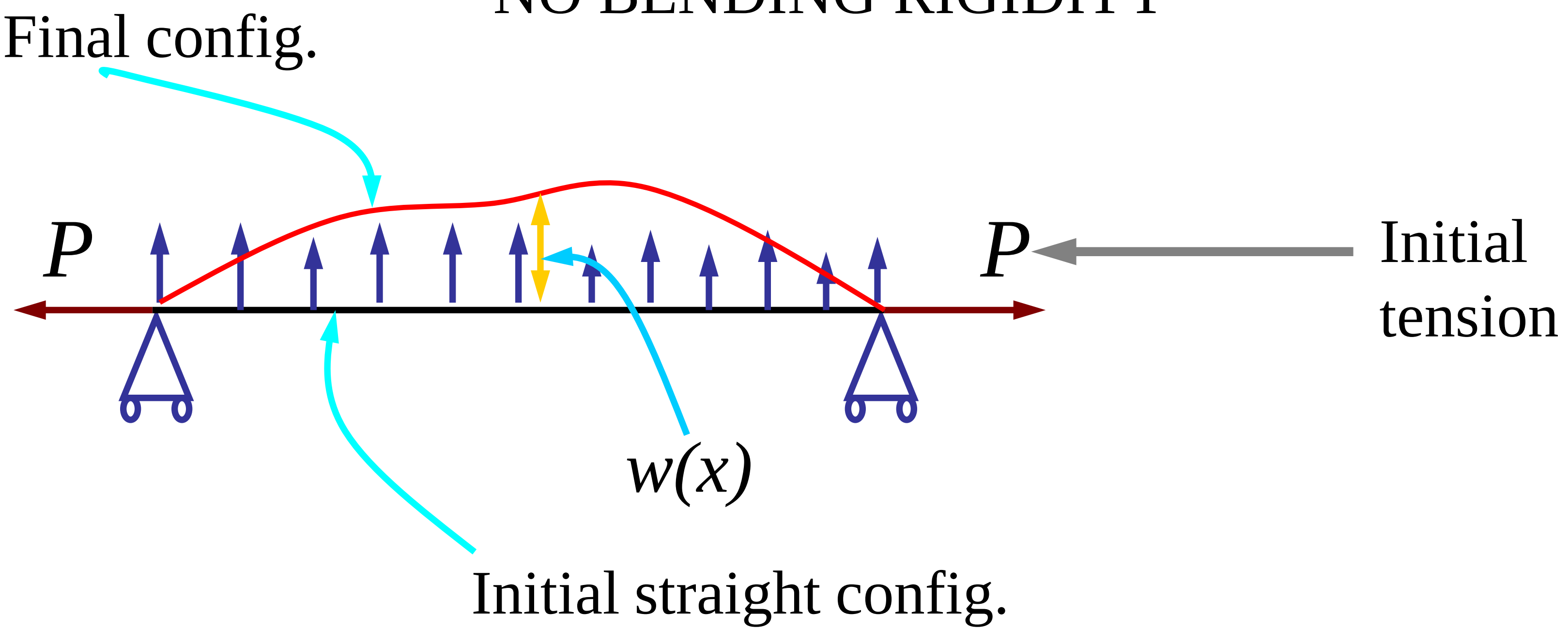
$$\delta^{(1)}\pi = \int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx}dx - \int_0^L f\delta u\,dx - P_L\delta u_L = 0$$

- NOTE THAT THE VARIATIONAL FORMULATION AND THE WEAK FORMULATION ARE THE SAME FOR THIS PROBLEM. THIS IS SPECIFIC TO THE SMALL DEFORMATION THEORY AND LINEAR PROBLEMS
- THE TWO WILL *NOT BE THE SAME* FOR NONLINEAR PROBLEMS, IN GENERAL (MATERIAL OR GEOMETRIC NONLINEARITY)
- THE PRINCIPAL OF VIRTUAL WORK IS THE MORE GENERAL FORMULATION BECAUSE IT COMES FROM THE CONSERVATION LAWS (WHICH ARE ASSUMED TO BE UNIVERSALLY TRUE)

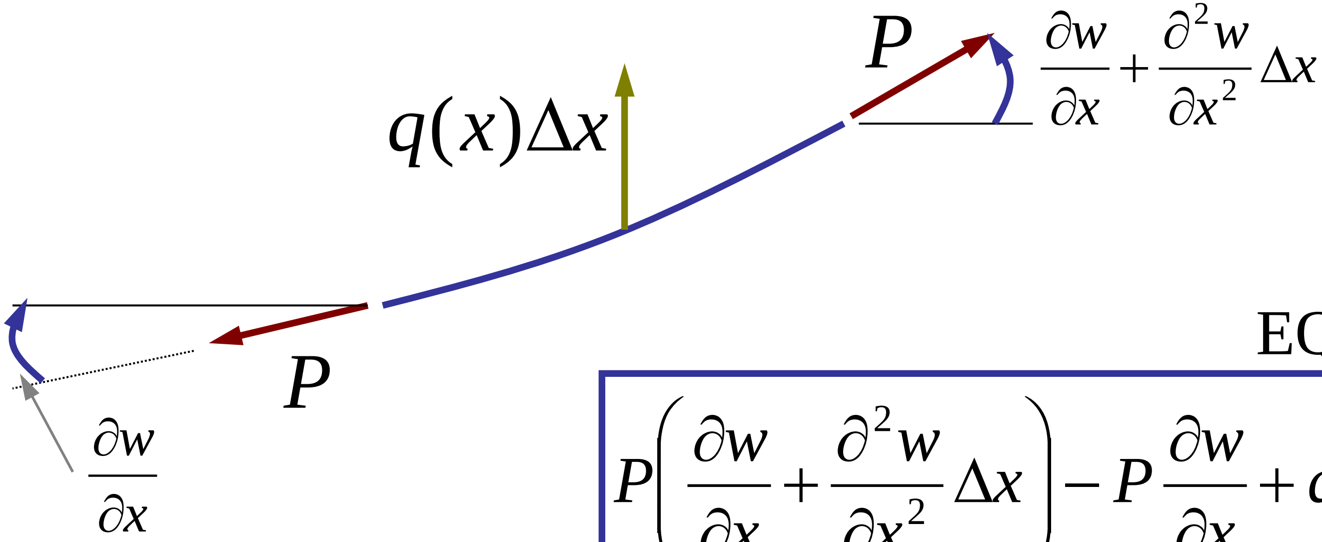
THE TWO-DIMENSIONAL MEMBRANE PROBLEM

(*ALSO THE STEADY-STATE HEAT CONDUCTION AND POTENTIAL FLOW PROBLEMS*)

NO BENDING RIGIDITY



STRINGS….



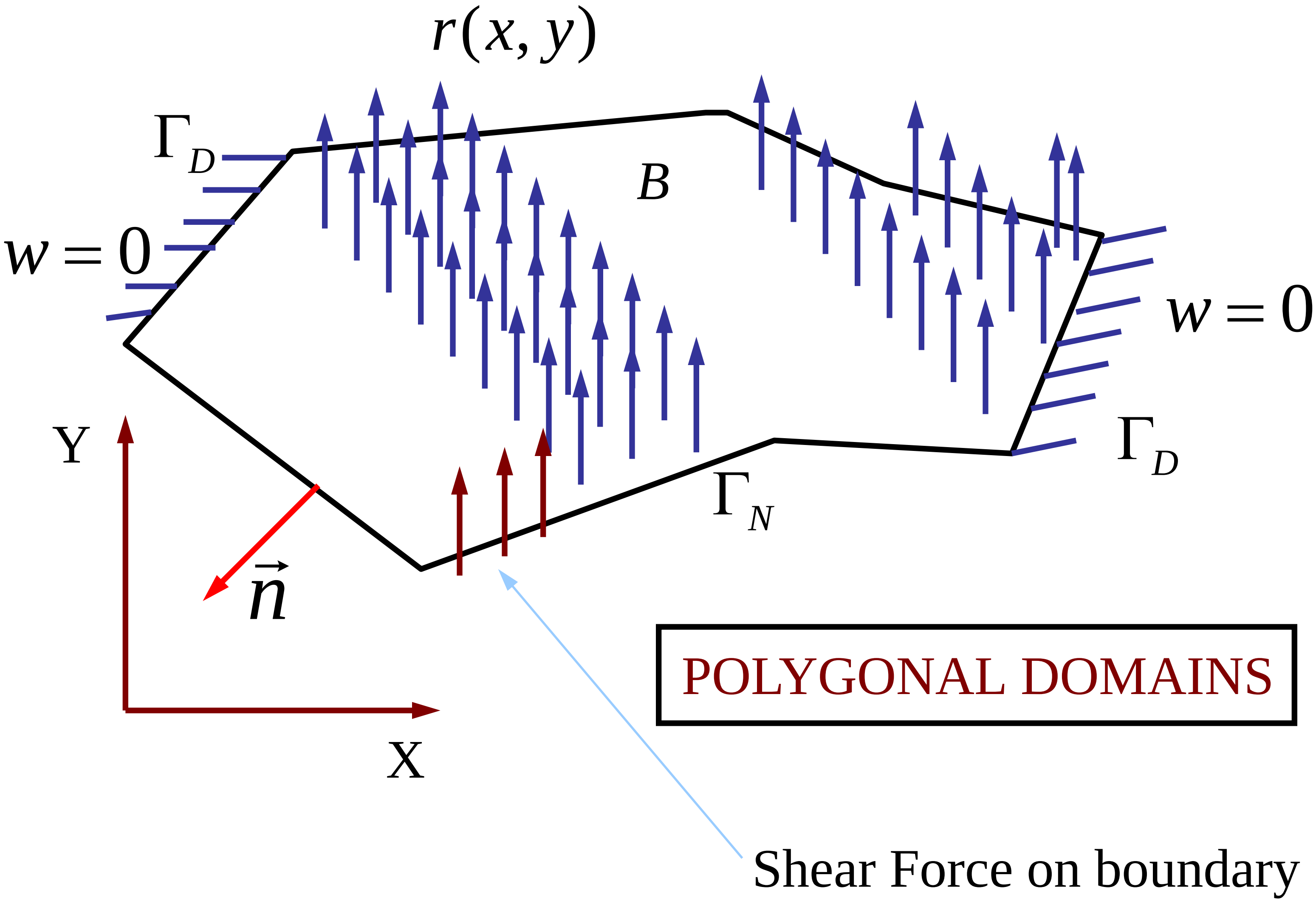
EQUILIBRIUM

$$P\left(\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}\Delta x\right)-P\frac{\partial w}{\partial x}+q\Delta x=0$$

$$\Rightarrow\ -P\frac{\partial^2 w}{\partial x^2}=q\quad\Rightarrow\quad -\frac{\partial^2 w}{\partial x^2}=\frac{q}{P}=r(x)$$

MEMBRANES

$$-\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=r(x,y)$$



POLYGONAL DOMAINS

THE RESIDUAL

$$R(x,y) = r(x,y) + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0$$

$$\int_B R(x,y)v(x,y)dA = \int_B \left(r(x,y) + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) v(x,y)dA = 0$$

NEED WEAKENING

$$\frac{\partial^2 w}{\partial x^2} v = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} v \right) - \frac{\partial w}{\partial x}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 w}{\partial y^2} v = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} v \right) - \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}$$

$$\int_B \left(\frac{\partial^2 w}{\partial x^2} \right) v \, dA = \int_B \left(\frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} v \right) - \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} \right) dA$$

$$\int_\Gamma \frac{\partial w}{\partial x} v n_x \, dS$$

Gauss-Green Divergence theorem

$$\int_B \left(\frac{\partial^2 w}{\partial y^2} \right) v \, dA = \int_B \left(\frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} v \right) - \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} \right) dA$$

$$\int_\Gamma \frac{\partial w}{\partial y} v n_y \, dS$$

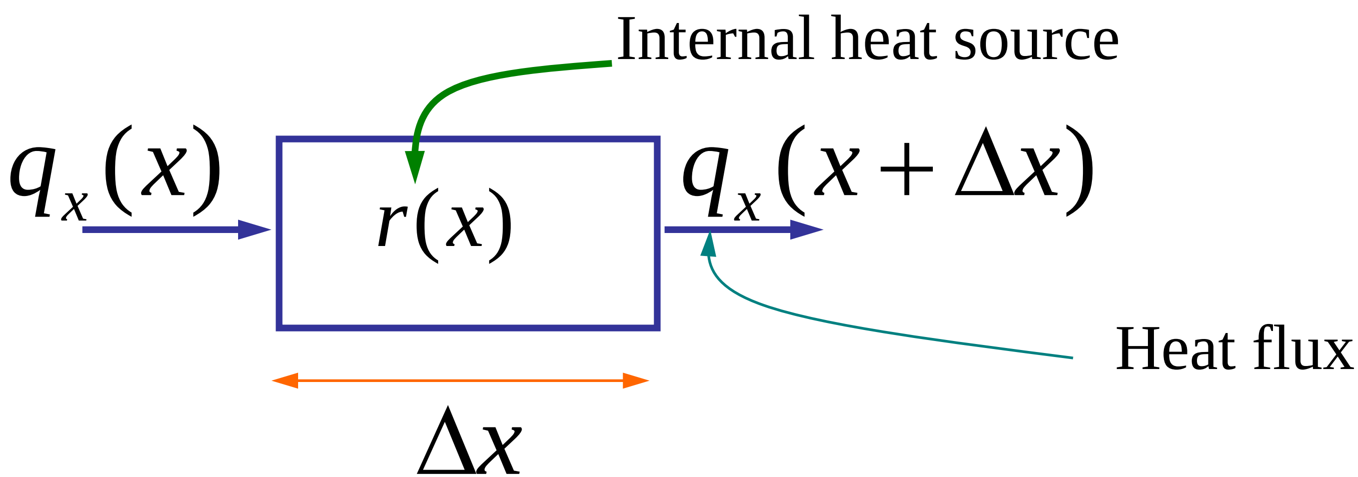
$$\int_B \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} \right) dA = \int_B (rv)\, dA + \int_{\Gamma_D + \Gamma_N} \underbrace{\left(\frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \right)}_{\frac{\partial w}{\partial n}} v\, dS$$

$$= \int_{\Gamma_N} \frac{\partial w}{\partial n} v\, dS$$

$$v = 0 \quad \text{on} \quad \Gamma_D$$

THE WEAK FORMULATION

EXTRA STUFF… HEAT CONDUCTION IN 1D



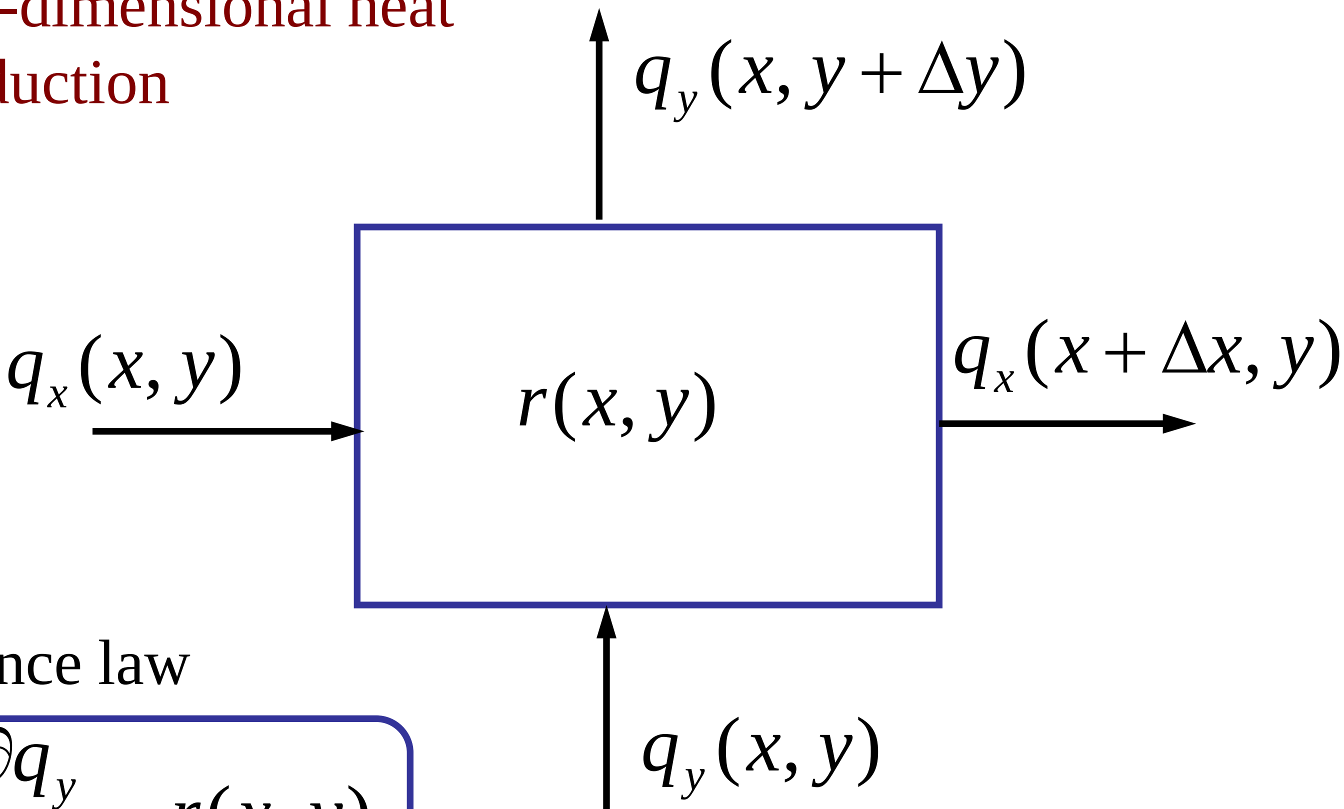
$$q_x(x+\Delta x) - q_x(x) - r(x)\Delta x = 0$$

$$\Rightarrow \boxed{\frac{dq_x}{dx} - r(x) = 0}$$

Constitutive relation

$$\boxed{q_x = -k\frac{dT}{dx}}$$

Two-dimensional heat conduction



The balance law

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = r(x, y)$$

The constitutive relation

$$q_x = -(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y})\,; \quad q_y = -(k_{12}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y})$$

- The unknown variable is the temperature T
- The equation is a generalization of the membrane problem
- The boundary conditions will also turn out to be similar, i.e. either the temperature (Dirichlet) or the normal heat flux is given at the boundary (Neumann).
- The weak form follows by exactly the same procedure as that for the membrane problem
- Note that in the case of radiative boundary condition, the boundary condition becomes NONLINEAR in terms of the T at the boundary (like the mixed B.C. problem in 1D – with a nonlinear spring)
- Note that here we CANNOT define a potential energy functional – makes no sense.

• A similar derivation also follows for the torsion of a non-circular cross-section, where the Prandtl stress function becomes the unknown to be solved for.

•For potential flow problems, the unknown function becomes the flow potential (stream function?).

•This is nice, as the same formulation can be used to solve four (and many more) completely disparate physical problems. All one has to do is pass suitable material, boundary and loading data.

•This is what most codes are exploiting in their new "multi-physics" version. Since the element stiffness has the same structure, it is computed symbolically and stored in a generic way. The actual values are obtained by substituting numbers – very fast explicit computations !!!