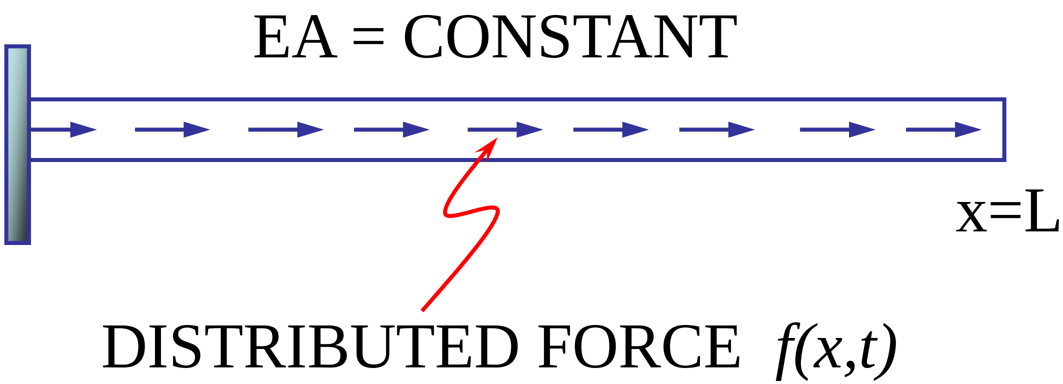
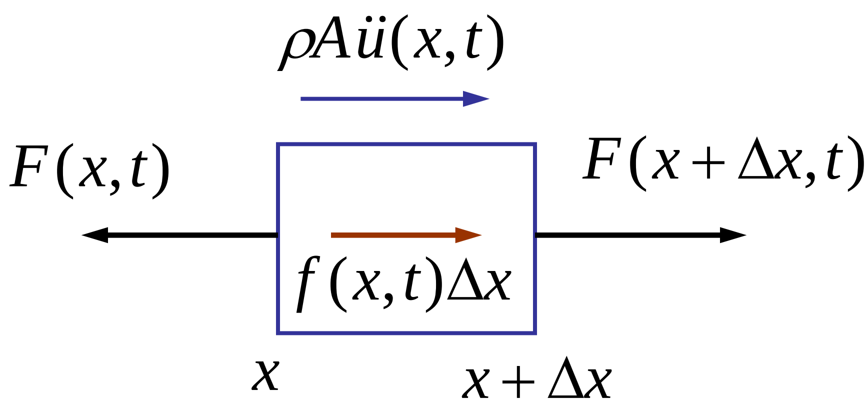
FREE VIBRATION OF A BAR – EIGENVALUE PROBLEM

- THE DYNAMIC BAR PROBLEM
- SEPARATION OF VARIABLES ANALYSIS
- FEM FORMULATION FOR THE EIGENMODES
- ACCURACY ESTIMATES
- STUDY OF WAVE EQUATION FOR GIVEN WAVE NUMBER – A DETAILED CASE STUDY





FREE-BODY DIAGRAM

EQUATION OF MOTION

$$\frac{dF}{dx} + f(x,t) = \rho A\ddot{u}$$

$$0 < x < L\,; \quad t > 0$$

INITIAL CONDITIONS:

$$u(x,0) = u_0(x); \quad \dot{u}(x,0) = v_0(x)$$

BOUNDARY CONDITIONS:

$$u(0,t) = 0; \quad EA\frac{du}{dx}\Big|_{x=L} = 0$$

UNDAMPED MOTION OF THE BAR

FREE VIBRATION PROBLEM ⇒ DISTRIBUTED FORCE

f(x,t)=0

•THE COMPLETE DYNAMICAL PROBLEM IS CALLED AN *IBVP* PROBLEM

•THE FREE VIBRATION OF THE BAR IS CAUSED DUE TO THE INITIAL PERTURBATION GIVEN TO THE BAR (INITIAL DISPLACEMENT OR VELOCITY)

$$\frac{dF}{dx} = \rho A \ddot{u} \quad \Rightarrow \quad \frac{d}{dx}\left(EA \frac{du}{dx} \right) = \rho A \ddot{u}$$

USE SEPARATION OF VARIABLES

$$u(x,t) = U(x) F(t)$$

PUTTING THIS IN THE DIFFERENTIAL EQUATION

$$EA\frac{d^2U}{dx^2}F(t) = \rho A U(x)\ddot{F}(t)$$

GIVES

$$\frac{E}{\rho}\frac{\frac{d^2U}{dx^2}}{U} = \frac{\ddot{F}}{F} = -\omega^2$$

Negative to ensure that the solution does not blow up with time

Oscillatory motion with natural frequency ω

THE CONTINUOUS EIGENVALUE PROBLEM

$$\frac{E}{\rho}U'' + \omega^2 U = 0 \qquad 0 < x < L$$

$$U(0) = 0; \quad U'(L) = 0$$

ASSUME $U(x) = e^{i\alpha x}$

$$-\frac{E}{\rho}\alpha^2 + \omega^2 = 0 \quad \Rightarrow \quad \alpha = \pm\sqrt{\frac{\rho}{E}}\omega = \pm k\omega$$

$$U(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

FROM THE B.C. AT x=0 $\quad U(0) = 0 \;\Rightarrow\; A = 0$

FROM B.C. AT x=L

$$U'(L) = 0 \;\Rightarrow\; \cos(\alpha L) = 0$$

$$\Rightarrow\; \alpha_n L = (2n-1)\frac{\pi}{2}; \quad n = 1,2,\cdots,\infty$$

THE nth MODE SHAPE (EIGENFUNCTION) IS GIVEN BY

$$U_n(x) = \sin(\alpha_n x) = \sin\left((2n-1)\frac{\pi x}{2L}\right)$$

ALSO…

$$\Rightarrow \omega_n = \sqrt{\frac{E}{\rho}}\alpha_n$$

nth NATURAL FREQUENCY

THE COMPLETE SOLUTION IS GIVEN BY

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\omega_n t + D_n \sin\omega_n t\right)\sin(\alpha_n x)$$

C_n , D_n OBTAINED FROM INITIAL CONDITIONS

- GENERALLY, ONE IS INTERESTED IN ONLY THE FIRST FEW NATURAL FREQUENCIES AND CORRESPONDING MODE SHAPES.
- CANNOT BE EXPLICITLY OBTAINED IN MOST CASES, WHERE THE SPATIAL DIFFERENTIAL EQUATION IS NOT EASY TO SOLVE.
- NOTE THAT HERE THE MODES ARE DISTINCT. IN PRACTICAL SYSTEMS, THE MODES CAN BE REPEATING OR CLUSTERED AROUND A GIVEN FREQUENCY.
- IN ACOUSTIC PROBLEMS, MODE SHAPES FOR SPECIFIC FREQUENCY BAND IS REQUIRED.

HOW TO USE FEM TO SOLVE FOR THIS?

START WITH THIS…..

$$EA\,U'' + \omega^2\,\rho A\,U = 0 \qquad 0 < x < L$$

$$U(0) = 0; \quad (EAU')\,|_{x=L} = 0$$

GET WEIGHTED RESIDUAL FORMULATION

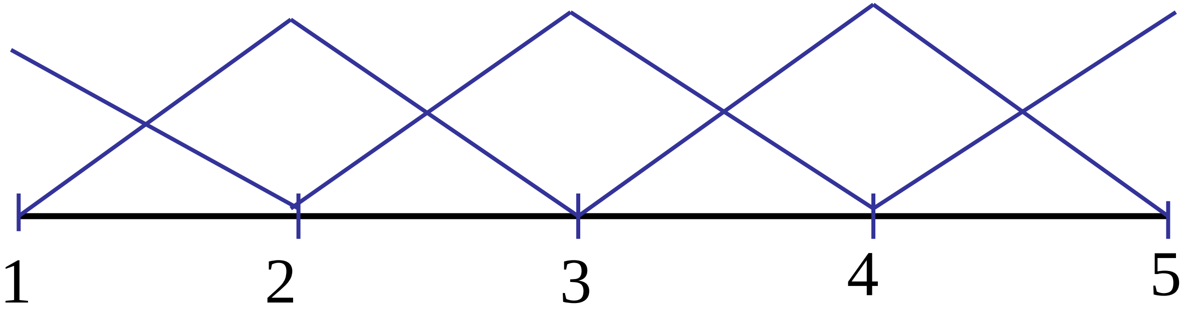
$$r(x) = EAU'' + \omega^2\,\rho A U = 0$$

$$\int_{x=0}^{L} r(x) w(x) dx = 0 \Rightarrow \int_{x=0}^{L} \left(EAU'' + \omega^2 \rho A\,U \right) w(x) dx = 0$$

INTEGRATE BY PARTS ONCE…

$$\int_{x=0}^{L} EA\, U' w' \, dx \;=\; \omega^2 \int_{x=0}^{L} \rho A U\, w\, dx \;+\; \left[EAU'\; w \right]_{x=0}^{x=L}$$

AGAIN C^0 ELEMENTS WILL WORK !



$$U(x) = \sum_{i=1}^{N} \delta_i\, \phi_i(x) \quad ; \quad w(x) = \sum_{i=1}^{N} \beta_i\, \phi_i(x)$$

THIS WILL GIVE THE STANDARD FORM

$$[K]\{\delta\} = \omega^2 [M]\{\delta\}$$

STIFFNESS MATRIX

MASS MATRIX

GENERALIZED EIGENVALUE PROBLEM

- THE FEM SOLUTION WILL APPROXIMATE THE FIRST *N* EIGENVALUES AND MODE SHAPES OF THE SOLUTION

- ONLY THE FIRST (APPROXIMATELY) *N/2* EIGENPAIRS ARE ACCURATE. ACCURACY DECREASES WITH INCREASE IN FREQUENCY

$$\omega_{FE_l} \geq \omega_l \qquad ; \text{ eigenpair } \omega_{FE_l}, \{\delta^{(l)}\}$$

The *l th* mode shape

$$U_{FE_l}(x) = \sum_{i=1}^{N} \delta_i^{(l)} \phi_i(x)$$

HOW ACCURATE IS THE SOLUTION?

$$|\omega_{FE_l} - \omega_l| \leq C_l \| U_l - U_{FE_l} \|^2$$

DEPENDS ON HOW WELL THE MODE SHAPE CAN BE REPRESENTED BY THE BASIS FUNCTIONS AND MESH

EXAMPLE TO SHOW USE OF FEM

$X_1 = 0$ $X_2 = 1$ $X_3 = 2$

$$U_{FE}(x) = \sum_{i=1}^{3} \delta_i \phi_i(x) \quad \Rightarrow \quad \delta_1 = 0$$

U(0)=0

THE MATRIX PROBLEM NOW TO BE SOLVED IS

$$\frac{EA}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{\delta\} = \left(\rho A h \omega^2\right)\begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\{\delta\}$$

Stiffness matrix

Mass matrix

SOLVING THIS EIGENVALUE PROBLEM GIVES

Material data taken $h = 1, \quad EA = 1, \quad \rho A = 1$

$$\omega_{FE_1} = 0.806\ (0.785) \qquad \omega_{FE_2} = 2.815\ (2.355)$$

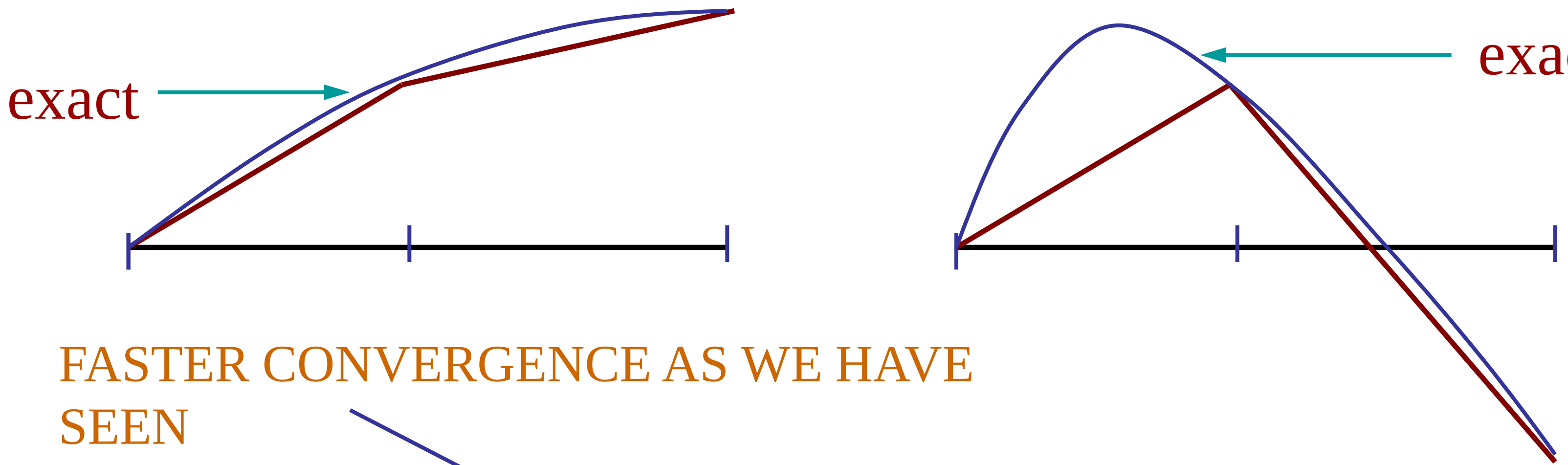
EXACT ONES IN PARENTHESIS

ERROR = 2.7% AND 16.34%

- NOTE THAT THE FEM VALUES ARE HIGHER!!!
- THE ERROR IN THE EIGENVALUE INCREASES AS THE EIGENVALUE NUMBER INCREASES
- LET US LOOK AT THE MODE SHAPES

$$\omega_{FE_1} \Rightarrow \quad \{\delta^{(1)}\} = \begin{Bmatrix} 0.707 \\ 1.000 \end{Bmatrix};\ \omega_{FE_2} \Rightarrow \quad \{\delta^{(2)}\} = \begin{Bmatrix} 0.707 \\ -1.000 \end{Bmatrix}$$

Note that these give exact nodal values of the actual mode shape



FASTER CONVERGENCE AS WE HAVE SEEN

A FINER MESH OR A HIGHER ORDER APPROXIMATION WILL GIVE A BETTER RESOLUTION OF THE MODES (AGAIN, HALF THE NUMBER WILL BE QUITE ACCURATE)

THE REVERSE PROBLEM ….

WHAT IF WE ARE INTERESTED IN OBTAINING THE MODE SHAPE OF A PARTICULAR FREQUENCY?

$$EA\,U'' + \omega^2 \rho A\,U = 0 \qquad 0 < x < 2$$

$$U(0) = 0; \quad U|_{x=2} = 1; \quad \omega = \frac{\pi}{4};$$

$$EA = 1 \qquad \rho A = 1$$

$$U_{FE}(x) = \sum_{i=1}^{3} \delta_i \phi_i(x)$$

⇐ FEM SOLUTION WITH SAME MESH

THE WEAK FORM IS THE SAME, i.e.

$$[K]\{\delta\} = \omega^2 [M]\{\delta\}$$

WITH

$$\delta_1 = 0 \qquad \delta_3 = 1$$

THIS GIVES $\delta_2 = 0.694$

NOT EXACT ANYMORE!!!

EXACT=0.707

THIS IS A FEATURE OF THESE COMPUTATIONS. THE NODAL FE VALUES ARE NOT EXACT. IN FACT, THE FE SOLUTION THINKS IT IS RIDING ON A SLOWER MOVING WAVE (NUMERICAL DISPERSION!!!) WITH FREQUENCY $\overline{\omega}$.

- ALL THE FEATURES OF THE FE SOLUTION SEEN IN THIS EXAMPLE HOLD FOR ALL THE MODELS/DOMAINS THAT YOU CAN COME ACROSS.
- IN TWO AND THREE DIMENSIONS, THE MODE SHAPES GET AFFECTED BY THE GEOMETRY OF THE DOMAIN AND HAVE THE SINGULAR (STRESS CONCENTRATION) BEHAVIOR THAT WE SAW IN THE ELASTICITY PROBLEM. HENCE, FOR THESE CASES ALSO, WE HAVE TO USE THE PROPERLY REFINED MESHES.
- THE NATURAL FREQUENCIES CONVERGE TO THE EXACT ONES FROM ABOVE.
- THE FREQUENCY OF THE WAVE THAT THE FE SOLUTION RIDES, CONVERGES TO THE EXACT ONE FROM BELOW.

THE ONE-DIMENSIONAL AXIAL WAVE PROBLEM

$$\frac{d^2u}{dx^2} + k^2 u = f = 1 \text{ (let's say)}$$

$$u(0) = u(L) = 0$$

UNIFORM MESH WITH MESH-SIZE h. (linear elements)

$$\cos(\bar{k}h) = \frac{1 - \frac{k^2h^2}{3}}{1 + \frac{k^2h^2}{6}}$$

FE SOLUTION CORRESPONDS TO ANOTHER WAVE WITH $\bar{k}$

$$h = \frac{\lambda}{n} = \frac{2\pi}{kn}$$

$$\bar{k} = k - \Delta k \qquad \delta = \frac{\Delta k}{k}$$

MESH SIZE GIVEN IN TERMS OF n ELEMENTS PER WAVELENGTH

- FE SOLUTION <u>TRAILS</u> THE EXACT SOLUTION
- THE NODAL VALUES ARE <u>NOT EXACT</u>

- FOR n=4, THE ERROR $\delta = 9\ \%$
- FOR n=5, THE ERROR $\delta = 5.6\ \%$
- FOR n=9, THE ERROR $\delta = 1.9\ \%$

} ERROR IN THE WAVE FREQUENCY (Dispersion)

- IN 2D OR 3D, THE EFFECT OF UNSMOOTHNESS OF GEOMETRY, AND MATERIAL BOUNDARIES, WILL MAKE THIS DEMAND MORE SEVERE !!!