OBJECTIVES

- MOTIVATION
- THE ONE-DIMENSIONAL PROBLEM
- ERROR IN SOLUTION FOR 1D SOLUTION
- TOOLS FOR A-POSTERIORI ESTIMATION OF ERROR
- EXAMPLES OF ERROR ESTIMATION AND ADAPTIVE ANALYSIS
- REMARKS

AN ACTUAL COMPONENT: ***TAIL ROTOR BLADE*** FOR A MINI-HELICOPTER

Sharp corners

Carbon cloth + epoxy

Attachment point

(Not so) Long and slender

Heterogeneous media

Actual component: Heterogeneous Continuum, complicated geometry, structural details, large loads (aerodynamic+inertia) + dynamics

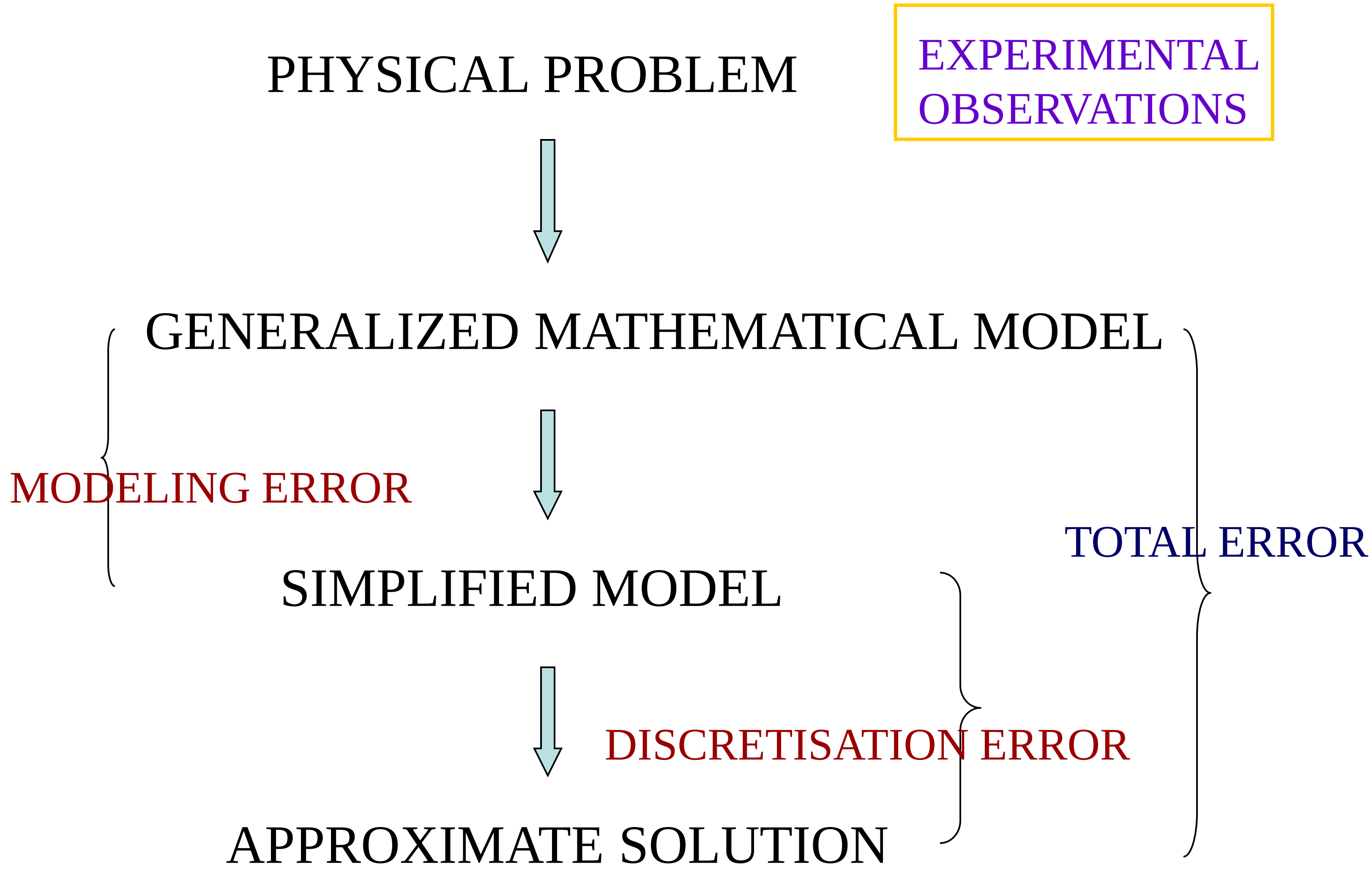
Uncertain loads, material, geometry

Equivalent homogeneous Continuum, CAD to IDEAS geometry (or point cloud to IDEAS)

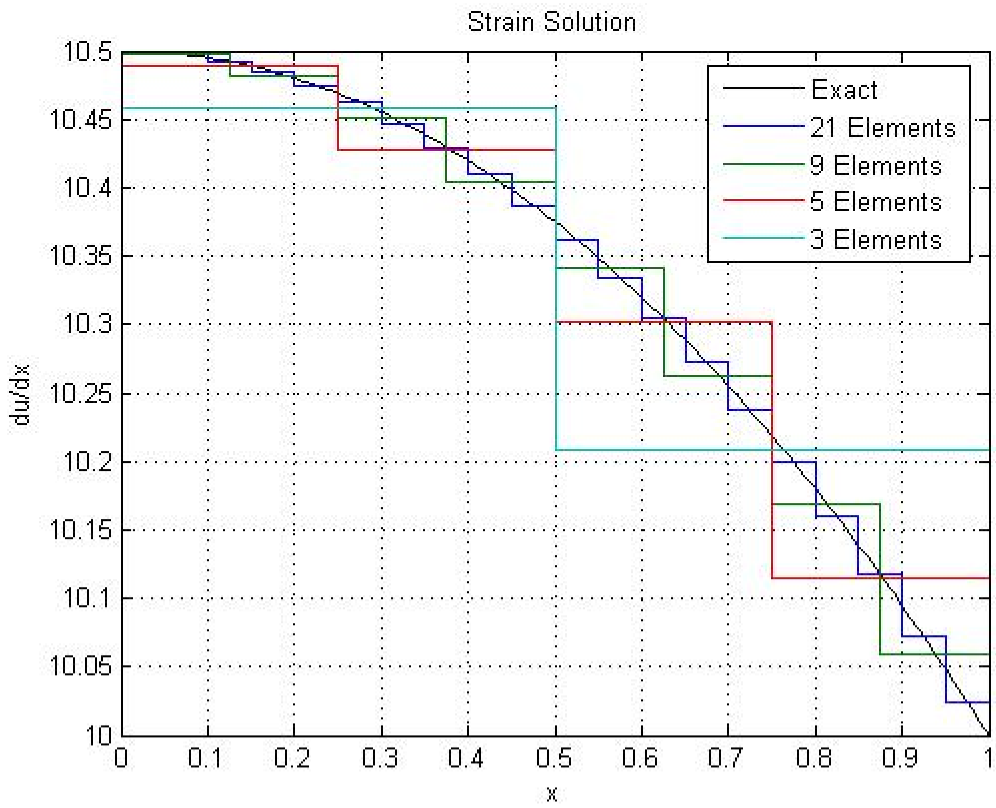
Idealized geometry, deterministic material

3D to quasi 1D (generalized beam + torsion + stretch) + Idealized boundary conditions + Equivalent static load +Linear elastic behavior (finite deformation case?)

SOLVE FOR THIS

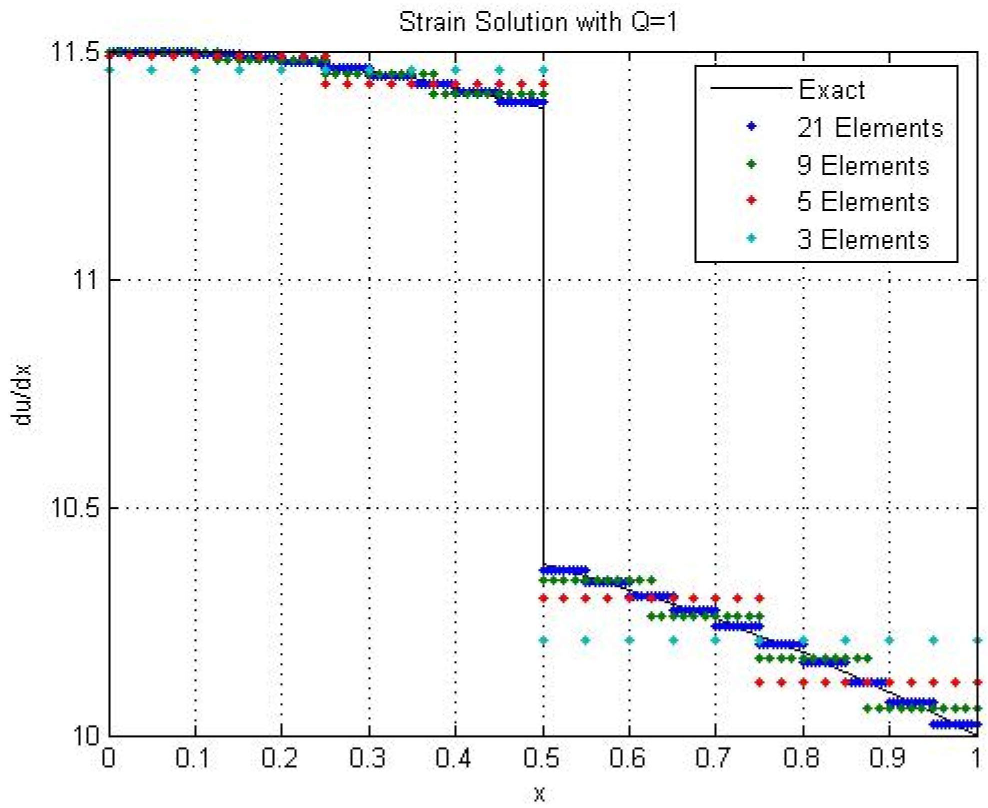

PHYSICAL PROBLEM
EXPERIMENTAL OBSERVATIONS
GENERALIZED MATHEMATICAL MODEL
MODELING ERROR
TOTAL ERROR
SIMPLIFIED MODEL
DISCRETISATION ERROR
APPROXIMATE SOLUTION

The outcome of the homework problems



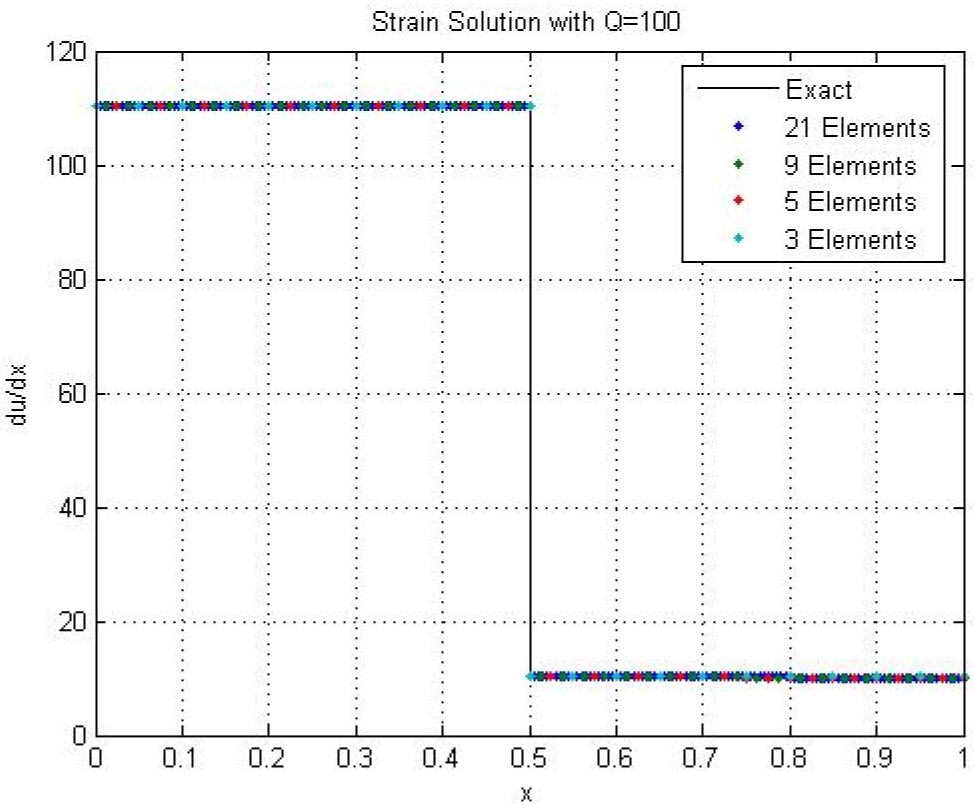
Uniform distributed load of intensity q=1

Point load and distributed load are fighting for supremacy



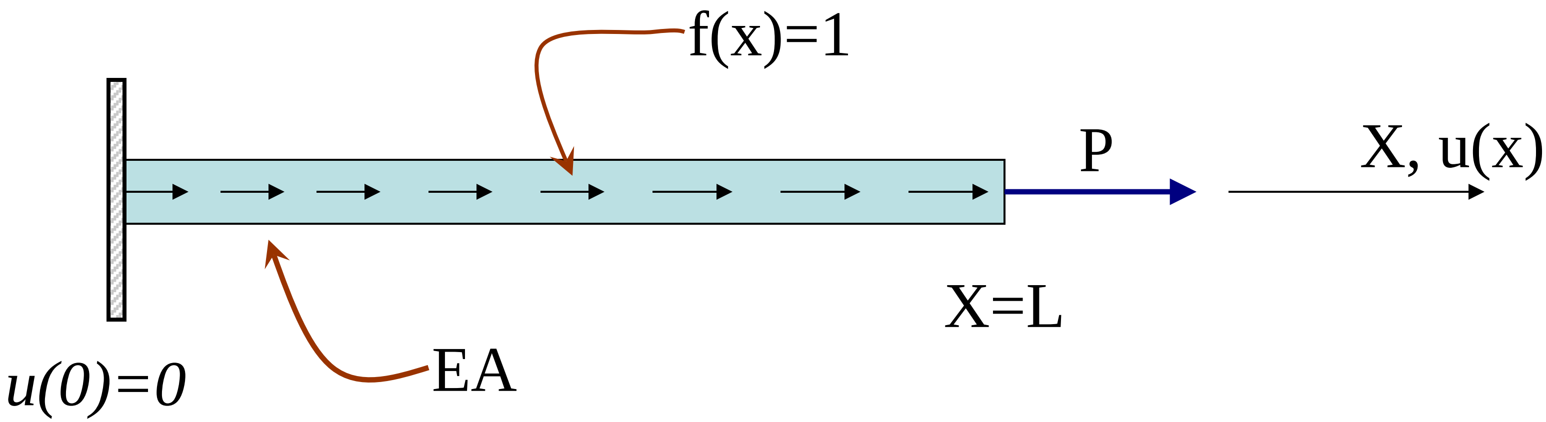
Q = 1

The point load dominates….



Q = 100

AXIAL BAR PROBLEM



$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x) \qquad 0 < x < L$$

WEAK FORMULATION (FE FORMULATION)

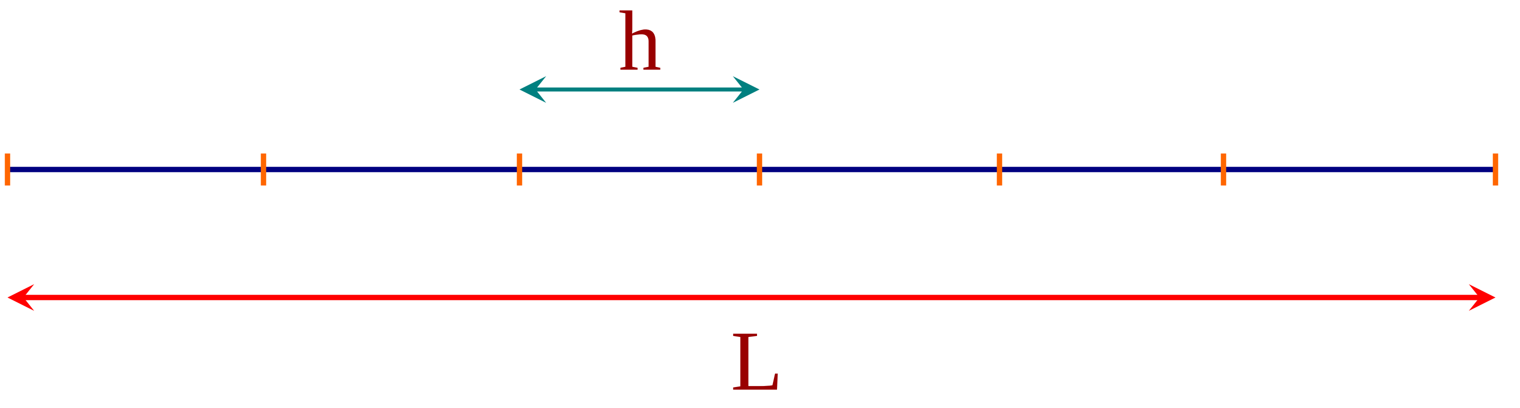
$$\int_{x=0}^{x=L} EA \frac{du}{dx}\frac{dv}{dx}\,dx = \int_{x=0}^{x=L} f\,v\,dx + P\,v(x=L)$$

$$B(u,v) \quad = \quad F(v)$$

$$\frac{1}{2}B(u,u) = U(u)$$

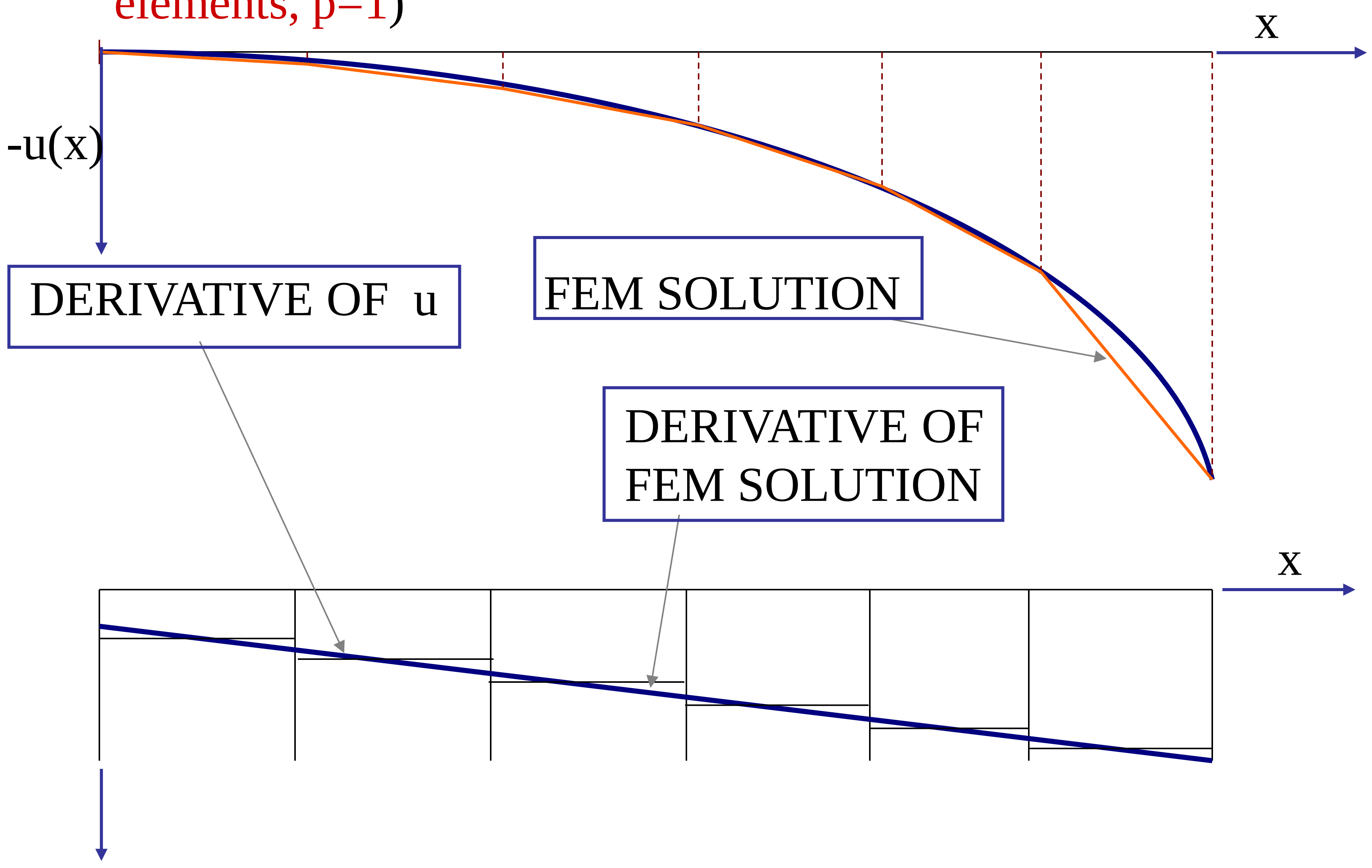
STRAIN ENERGY

THE MESH



- UNIFORM MESH WITH MESH-SIZE h
- NUMBER OF ELEMENTS $N = L/h$
- ELEMENT WITH STRAIN A POLYNOMIAL OF ORDER (p-1) ⇒ ELEMENT OF ORDER p.

THE EXACT AND FEM SOLUTION (constant strain elements, p=1)



AN EXAMPLE PROBLEM: *L, EA, f(x)=a, P=0*

PROBLEM SOLVED USING CONSTANT STRAIN ELEMENTS (p=1) AND mesh-size h

STRAIN ENERGY OF EXACT SOLUTION= $\frac{a^2 L^3}{6EA}$

STRAIN ENERGY OF FEM SOLUTION U(u_{FE}) IS

$$\frac{a^2 L}{6EA}\left(L^2 - \frac{h^2}{4}\right)$$

THIS LEADS TO

$$RE = \frac{\left(U(u) - U(u_{FE})\right)}{U(u)} = \frac{1}{4}\left(\frac{h}{L}\right)^2$$

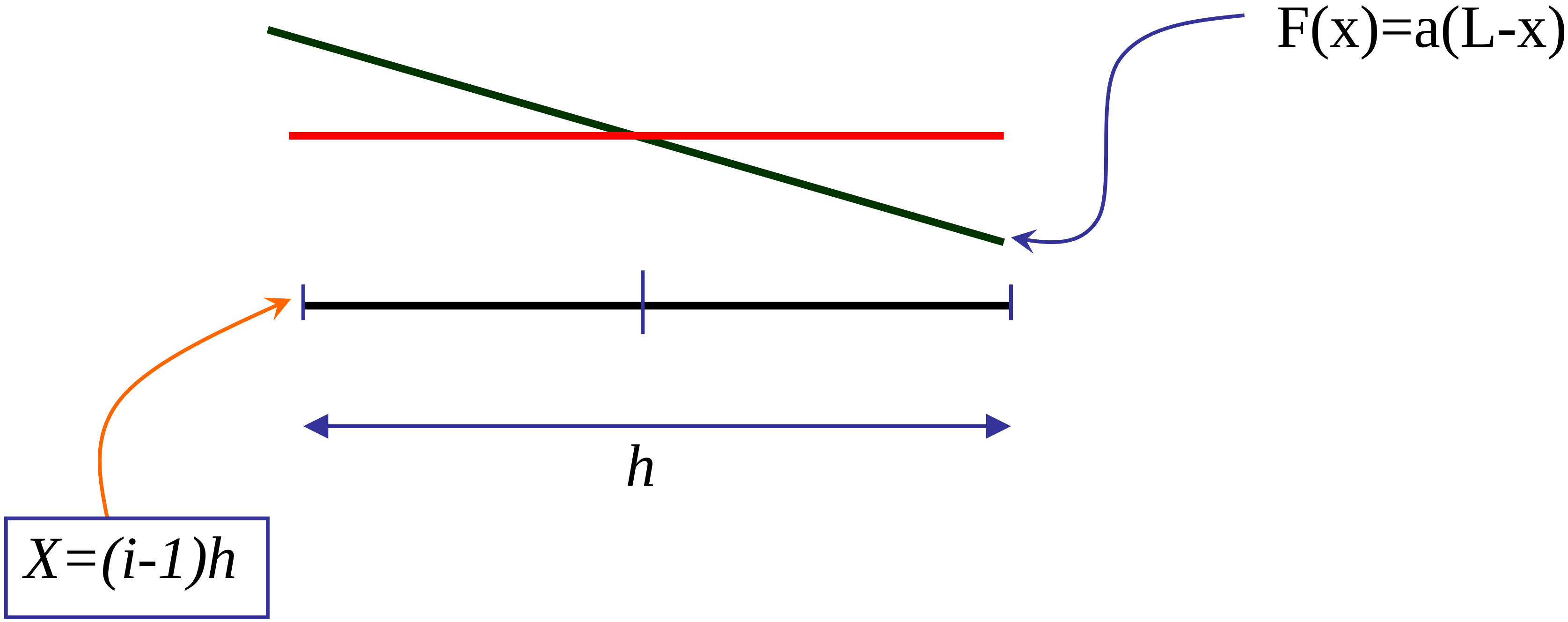
HERE, *RE* IS CALLED THE RELATIVE ERROR

FOR MESHES WITH 2,4,8 ELEMENTS, THIS GIVES

RE=1/16 (6.25%), 1/64 (1.56%), 1/256 (0.39%)

WHAT IS GOOD ENOUGH?

WHAT ABOUT THE AXIAL FORCE?



IN ELEMENT '*i*' THE AXIAL FORCE OBTAINED FROM FEM SOLUTION IS

$$F_{FE} = a(L-(X+0.5h))$$

NOTE SOME FEATURES OF THE FEM SOLUTION:

- THE AXIAL FORCE (OR AXIAL STRESS) IS "EXACT" AT THE CENTER OF THE ELEMENT
- THE AXIAL FORCE HAS MAXIMUM ERROR AT THE TWO ENDS OF THE ELEMENT, WITH THE ERROR GIVEN BY

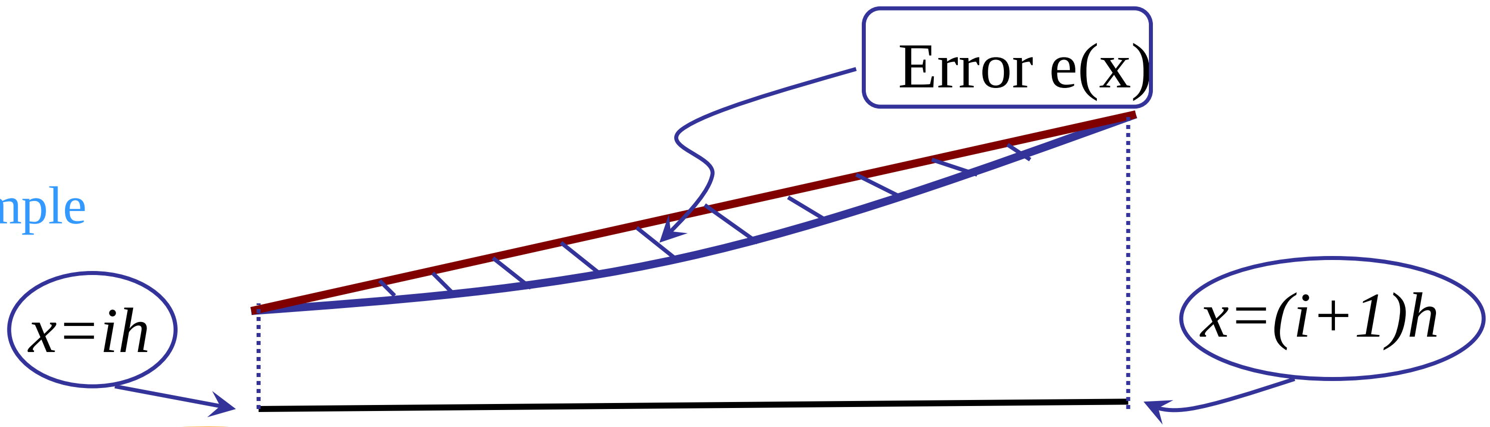
$$|F_{FE} - F(X)| = 0.5\, a\, h$$

- AS A RATIO, THIS ERROR IS $0.5 / (N - i + 1)$
- AT THE ROOT THIS IS, 25% (N=2), 12.5% (N=4), 6.25% (N=8)
- $\boxed{\sqrt{RE}}$ WOULD HAVE GIVEN SAME RESULT!!
- ERROR IN THE INTERNAL LOAD IS *a*. This is HUGE too!!

THE ERROR PROBLEM ⟺ RESIDUAL PROBLEM

• THE ERROR $(u - u_{FE})$ SATISFIES THE FOLLOWING PROBLEM:

a for example



$$-\frac{d}{dx}\left(EA\frac{de}{dx}\right) = f(x) - \frac{d}{dx}\left(EA\frac{du_{FE}}{dx}\right)$$ in the interior of element

ah for example

With jump $$J = EA\left(\frac{du_{FE}}{dx}\Big|_{x=ih-} - \frac{du_{FE}}{dx}\Big|_{x=ih+}\right)$$ at node i

- DO NOT WANT TO SOLVE THE ERROR PROBLEM
- HOW TO CONSTRUCT AN (GOOD) APPROXIMATION TO THIS?

- *A-POSTERIORI* ERROR ESTIMATION TECHNIQUES
- BASIC IDEA IS TO QUICKLY OBTAIN INFORMATION ABOUT ACCURACY OF THE SOLUTION BY SOLVING SMALL (INEXPENSIVE) PROBLEMS

- SEVERAL BROAD CLASSES OF A-POSTERIORI ERROR ESTIMATORS EXIST

HOW TO DO A-POSTERIORI ERROR ESTIMATION?

THREE BASIC TYPES OF ERROR ESTIMATORS

1. EXTRAPOLATION
2. RESIDUAL
3. AVERAGING BASED

EXTRAPOLATION BASED ESTIMATOR

DECAY OF ERROR

$$U(e_{FE}^{h}) = U(u) - U(u_{FE}^{h}) \approx \bar{C} h^{2\beta}$$

SEQUENCE OF SOLUTIONS

$$U(u) = A; \quad U(u_{FE}^{h}) = B; \quad U(u_{FE}^{h/2}) = C; \quad U(u_{FE}^{2h}) = D$$

THE UNKNOWNS

$$A = \frac{(B^2 - CD)}{(2B - C - D)}; \qquad \beta = \frac{1}{2\log 2} \log\left(\frac{A-B}{A-C}\right)$$

$\bar{C}$ CAN BE FOUND FROM THE FIRST EQN.

FOR THE ELASTIC BAR PROBLEM, USING THE SOLUTIONS FOR N = 2,4,8 , ONE GETS

$$A = U(u) = \frac{a^2 L^3}{6EA}$$

$$\overline{C} = \frac{a^2 L}{24EA}$$

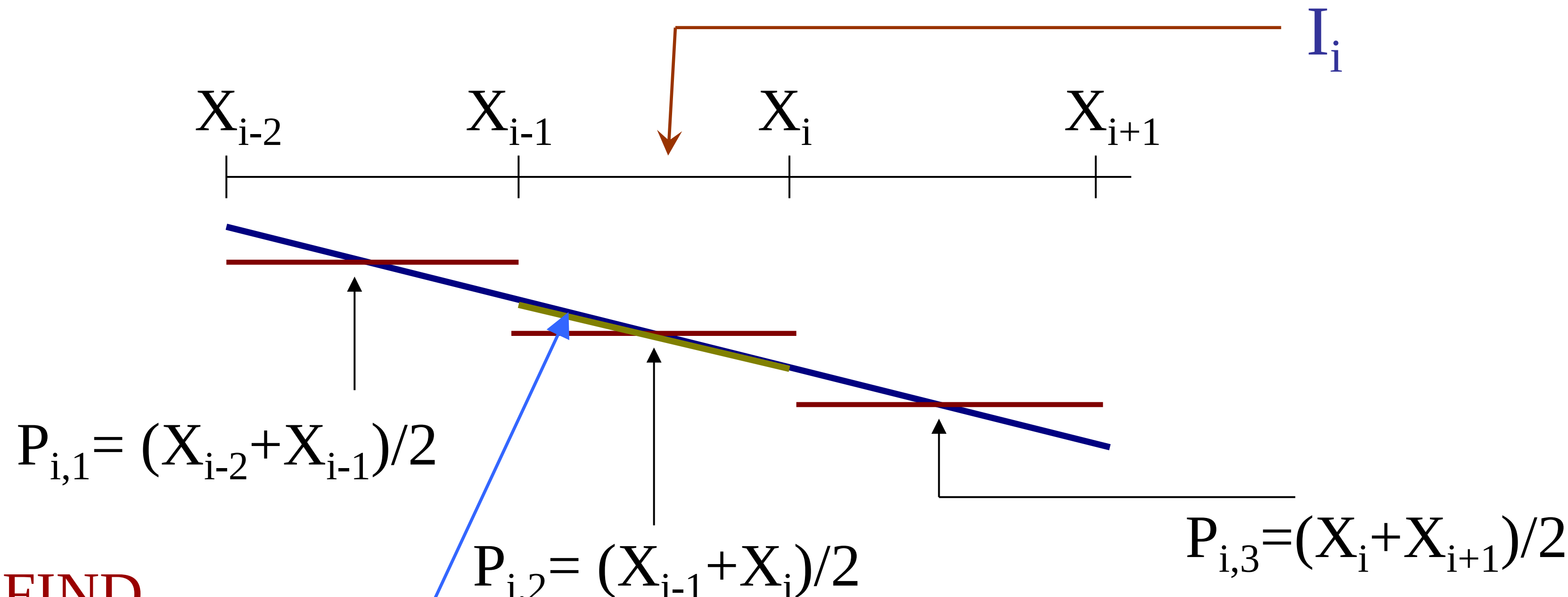
$$\beta = 1$$

EXACT!!!

•IN GENERAL, VERY EFFECTIVE WHEN SOLUTIONS OVER MULTIPLE MESHES AVAILABLE

AVERAGING BASED ERROR ESTIMATOR

(ZZ-TYPE ESTIMATORS)



FIND

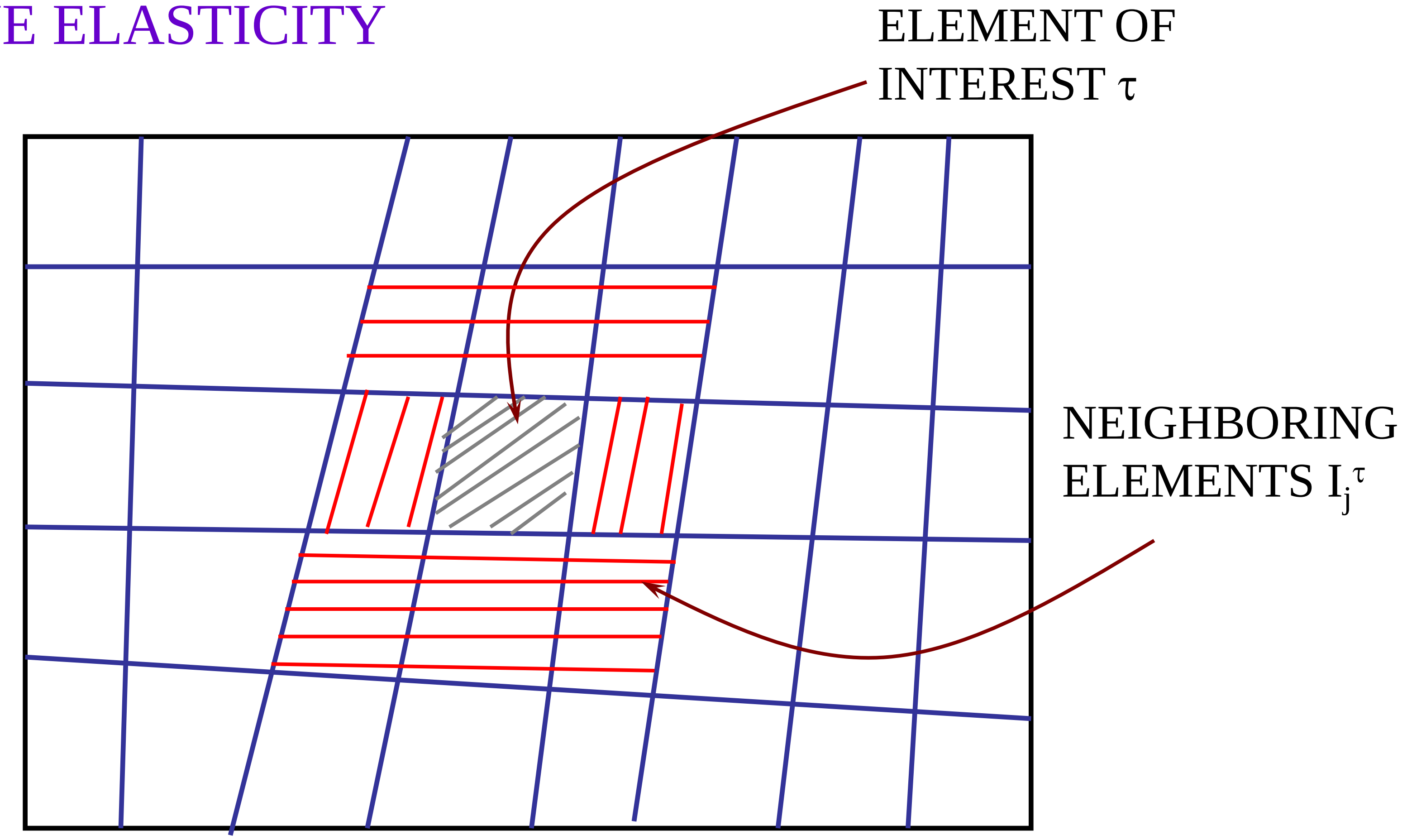
$$\frac{du^*}{dx}\bigg|_{I_i} = a_0^i + a_1^i x$$

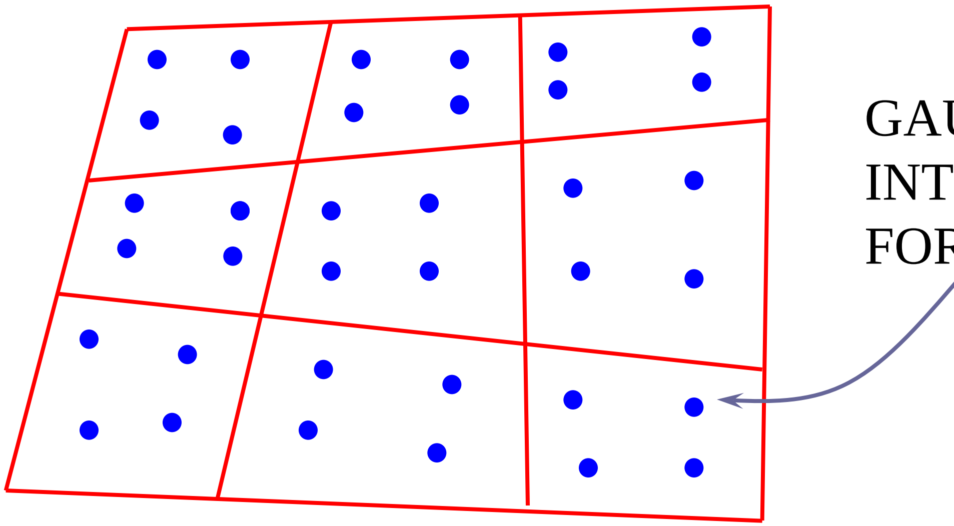
$$J^{(i)} = \sum_{j=1}^{3} EA\left(\frac{du^*}{dx}\bigg|_{P_{i,j}} - \frac{du_{FE}}{dx}\bigg|_{P_{i,j}}\right)^2$$

J IS MINIMIZED

TWO DIMENSIONAL VERSION OF ZZ-ESTIMATOR

PLANE ELASTICITY





- TAKE ELEMENTS CONNECTED TO THE ELEMENT OF INTEREST

- ASSUME THAT EACH STRESS COMPONENT IN THE THIS REGION IS GIVEN BY A POLYNOMIAL OF THE ORDER p

$$\begin{Bmatrix} \sigma^*_{xx} \\ \sigma^*_{yy} \\ \sigma^*_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & x^2 & 0 & 0 & xy & 0 & 0 & y^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & x^2 & 0 & 0 & xy & 0 & 0 & y^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & x^2 & 0 & 0 & xy & 0 & 0 & y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{18} \end{Bmatrix}$$

Recovered stress components. Need to obtain the constants.

Minimize:

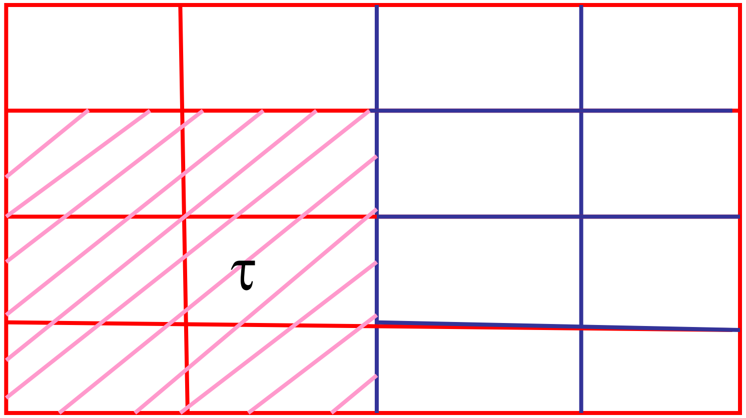
$$J^\tau = \frac{1}{2} \sum_{I_k^\tau} \sum_{i=1}^{NINT} \left\{ \sigma^* - \sigma_{FE} \right\}^T [C]^{-1} \left\{ \sigma^* - \sigma_{FE} \right\} \Big|_{\xi_i^k, \eta_i^k}$$

This will lead to an 18X18 system to be solved in terms of the 18 unknown coefficients.

Using the coefficients, construct the polynomial representations of the recovered stress components. Now define, for the element τ, the strain energy of error as

$$U_\tau^* = \frac{1}{2} \int_\tau \left\{ \sigma^* - \sigma_{FE} \right\}^T [C]^{-1} \left\{ \sigma^* - \sigma_{FE} \right\} dA$$

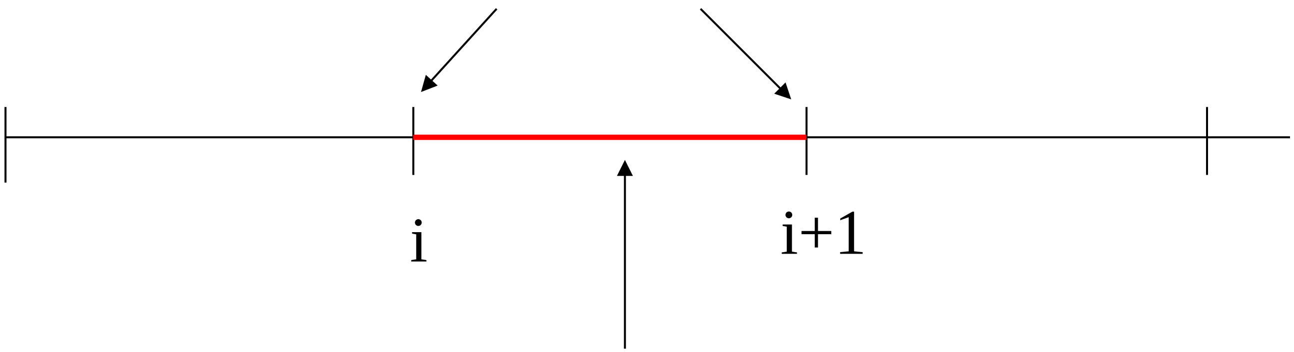
- These coefficients are used to construct the "recovered" stress field only in the element τ
- For each element, the process has to be repeated and an 18X18 problem has to be solved *NEL* times.
- At the boundary, we do the same job.
- At material interfaces, we take only the elements lying in the same material region.



SUBDOMAIN RESIDUAL ERROR ESTIMATOR

$$J_i = EA\left(\frac{du_{FE,i}}{dx} - \frac{du_{FE,i+1}}{dx}\right)$$

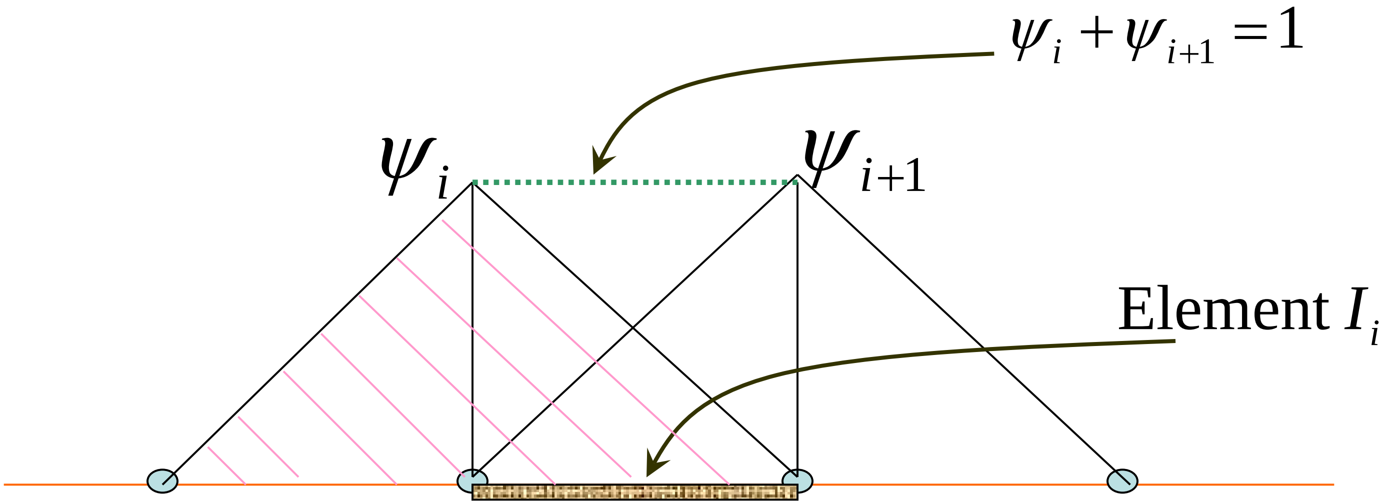
EDGE
JUMP



$$-\frac{d}{dx}(EA\frac{de}{dx}) = f - \frac{d}{dx}(EA\frac{du_{FE,i}}{dx}) = r$$

INTERNAL
RESIDUE

SOLVE ELEMENTWISE RESIDUAL PROBLEMS ….



$$B(e,v) = R(v) \quad \Rightarrow \quad B(\sum_{i=1}^{N} \psi_i \, e, v) = R(\sum_{i=1}^{N} \psi_i v)$$

$$\Rightarrow \quad \sum_{i=1}^{N} B(\tilde{e}_i, v) = \sum_{i=1}^{N} R(\psi_i v)$$

Partition of Unity Approach

SOLVE FOR:

$$B(\tilde{e}_i, v) \;=\; F(\psi_i v) \qquad \tilde{e}_i = \psi_i e$$

Both in higher order polynomial set

$$e_i = \psi_i \tilde{e}_i + \psi_{i+1} \tilde{e}_{i+1}$$

- FOR THE BAR PROBLEM, ALL THREE ESTIMATORS ARE EXACT (IN THE STRAIN ENERGY SENSE)
- IN GENERAL, ZZ ESTIMATOR IS VERY ROBUST AND COMPUTATIONALLY MOST ECONOMICAL
- THE SUBDOMAIN RESIDUAL LEADS TO GUARANTEED UPPER ESTIMATES OF THE ERROR (AND LOWER ESTIMATES TOO)
- ELEMENT-BY-ELEMENT ERROR MAPS CAN BE OBTAINED, WHICH CAN BE USED TO REFINE MESH IN REGIONS OF HIGH ERROR (*ADAPTIVE ANALYSIS*)
- MESHING WITH RESPECT TO QUANTITY OF INTEREST CAN ALSO BE DONE