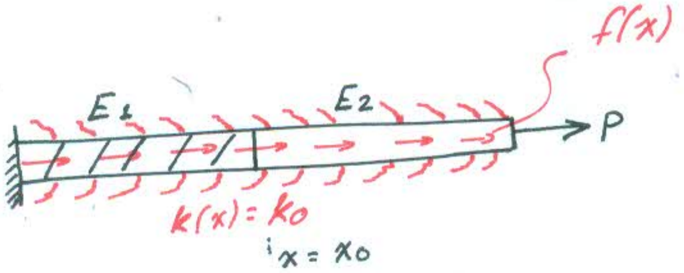
TRANSITION TO FEM

If the polynomial series does the job, why talk of anything else?

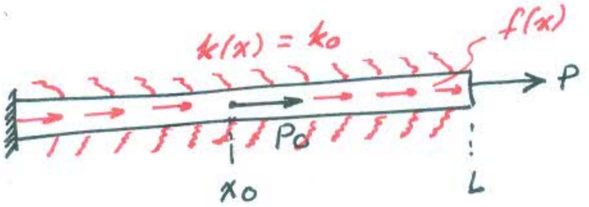
Let us see the reason for moving on:

(a)



Different material

(b)

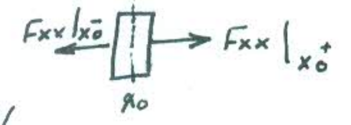


Point Load

Case (a): What happens at x_0?

You know that the differential equation (of equilibrium) is valid for $(0, x_0)$ and (x_0, L), i.e. at $x = 0, x_0, L$ the differential equation is NOT valid. Note that at $x = 0, L$ boundary conditions are PRESCRIBED; at $x = x_0$ we have to enforce continuity conditions OR

(1) $\boxed{u\big|_{x_0^+} = u\big|_{x_0^-}}$ ← CONTINUITY OF VALUE AT x_0 &



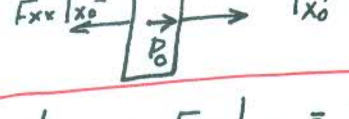
(2) $F_{xx}\big|_{x_0^+} = F_{xx}\big|_{x_0^-} \Rightarrow E_1 A\big|_{x_0} \frac{du}{dx}\Big|_{x_0^-} = E_2 A \frac{du}{dx}\Big|_{x_0^+}$

or $\boxed{\frac{du}{dx}\Big|_{x_0^-} = \left(\frac{E_2}{E_1}\right) \frac{du}{dx}\Big|_{x_0^+}}$ JUMP IN DERIVATIVE AT x_0

Case (b): Again at $x = x_0$ we have:

(1) $u\big|_{x_0^+} = u\big|_{x_0^-}$

(2)

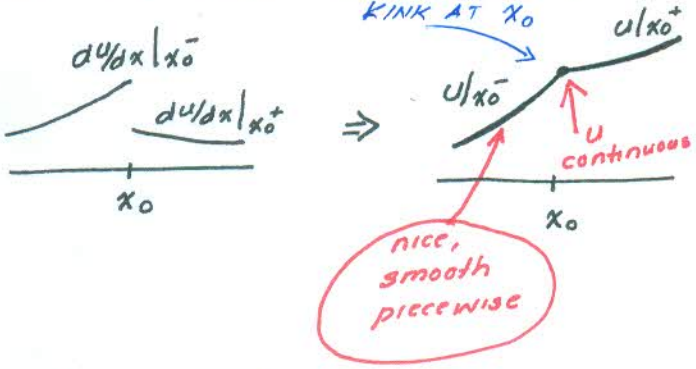


AXIAL FORCE JUMPS at x_0

$\Rightarrow \frac{du}{dx}\Big|_{x_0^-} - \frac{du}{dx}\Big|_{x_0^+} = \frac{P_0}{EA(x_0)}$ ⇒ $\boxed{F_{xx}\big|_{x_0^-} - F_{xx}\big|_{x_0^+} = P_0}$

Both cases are practically feasible, important and relevant.

If $\frac{du}{dx}$ jumps at a point

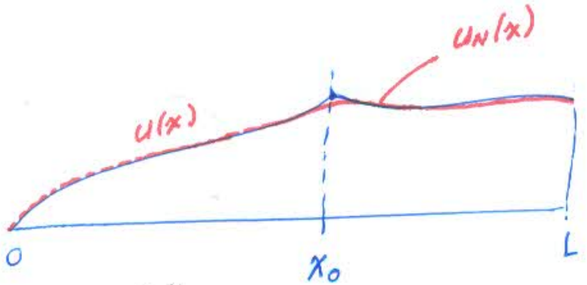


So, we CANNOT use the basis functions $\varphi_i(x) = x^i$, as

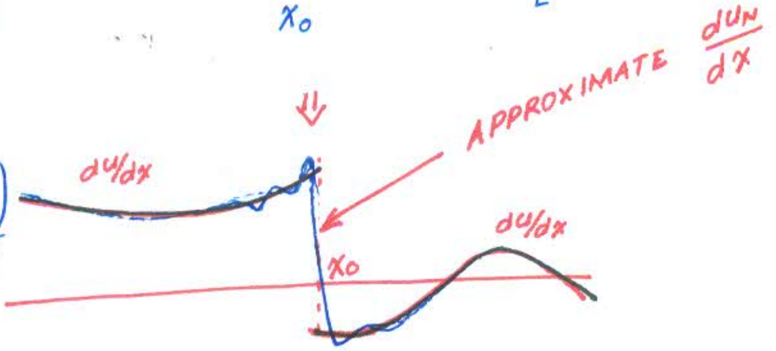
$$\frac{d\varphi_i}{dx} = i\,x^{i-1} \quad \text{which gives} \quad \left.\frac{d\varphi_i}{dx}\right|_{x_0^+} = \left.\frac{d\varphi_i}{dx}\right|_{x_0^-} = i\,x_0^{i-1}$$

and hence does not give the required jump in derivatives at $x = x_0$

WHAT IF I SOLVE WITH $\varphi_i(x) = x^i$ ANYWAY ?

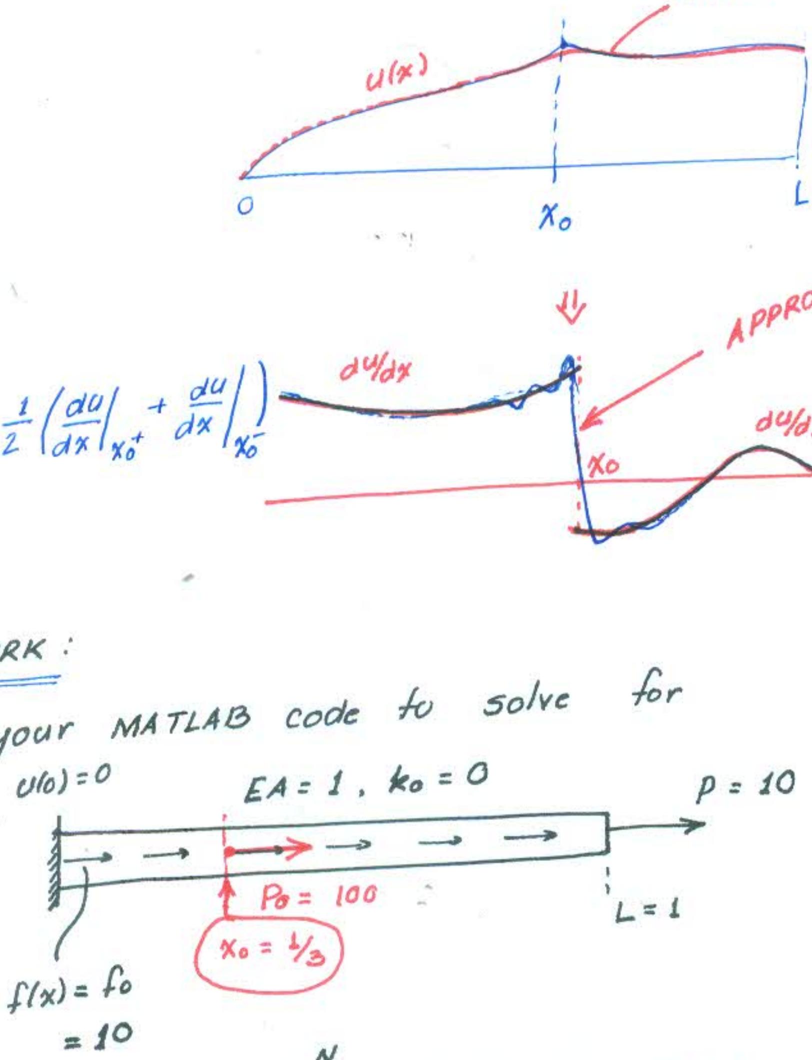


$$\underset{N\to\infty}{Lt}\ \frac{du_N}{dx} \rightarrow \frac{1}{2}\left(\left.\frac{du}{dx}\right|_{x_0^+} + \left.\frac{du}{dx}\right|_{x_0^-}\right)$$



HOME - WORK :

Use your MATLAB code to solve for



$$\boxed{\begin{aligned} F_i &= \int_0^1 10\cdot x^i\,dx \\ &+ P\cdot(L^i) + \\ &P_0\cdot(L/3)^i \end{aligned}}$$

ADDITIONAL PART IN LOAD VECTOR

With $u_N(x) = \sum_{i=0}^{N} d_i\,\varphi_i(x)$; $\varphi_i(x) = x^i$

USE $N = 2, 4, 8$

LESSON:

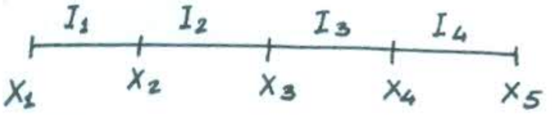
At $x = x_0$, the solution gives a wrong picture of $F_{xx}\Big|_{x_0}$ or $\frac{du}{dx}\Big|_{x_0}$. This is a problem as point load application points or MATERIAL transition points are CRITICAL stress points & the ANSWER we got does not help!

So, we need to create a better approximation. How?

$\boxed{\text{Piecewise polynomials?}}$

BASIC IDEA : 1) Divide domain $\Omega = (0, L)$ into some pieces. (i.e. NE pieces), as.

Piece $I_k = (x_k, x_{k+1})$



* Here No. of pieces, or ELEMENTS or SUBDOMAINS $NE = 4$

$NE = 4$

x_ℓ , $\ell = 1, 2, \ldots, (NE+1)$, are the NODES or VERTICES

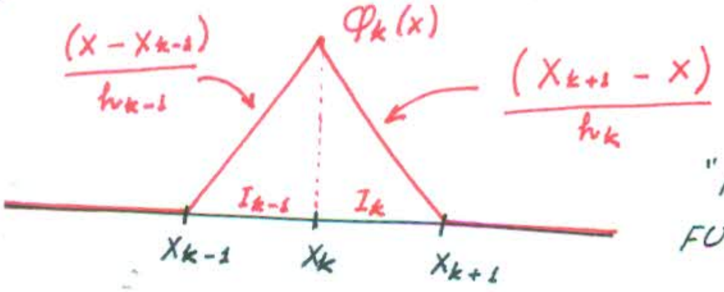
Finite (NE) no. of Elements

$I_k \rightarrow$ elements

2) Define the following functions, corresponding to each vertex x_k :

$h_{k-1} = x_k - x_{k-1}$

$h_k = x_{k+1} - x_k$



"HAT" FUNCTION

h_ℓ = length of I_ℓ = $(x_{\ell+1} - x_\ell)$ is also called mesh-size

The function $\varphi_k(x) = \begin{cases} \dfrac{(x - x_{k-1})}{h_{k-1}} & \text{in } I_{k-1} \\ \dfrac{(x_{k+1} - x)}{h_k} & \text{in } I_k \\ 0 & \text{everywhere else} \end{cases}$

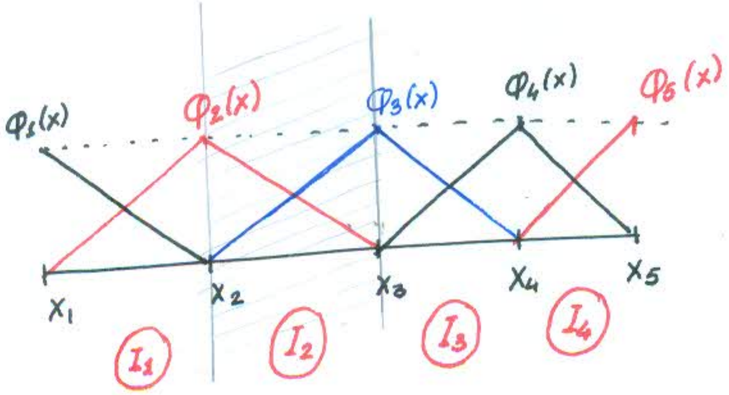
Note that (a) $\varphi_k(x)$ is defined in $\Omega = (0, L)$

(b) $\varphi_k(x)$ is non-zero in $I_{k-1} \cup I_k$ and is zero in rest of Ω ; $\varphi_k(x)$ is said to have LOCAL SUPPORT

(c) $\varphi_k(x)$ is LINEAR in I_{k-1} and I_k, i.e. it is piecewise linear.

(d) $\varphi_k(x_k) = 1$; $\varphi_k(x_{k-1}) = \varphi_k(x_{k+1}) = 0$

So, we have :



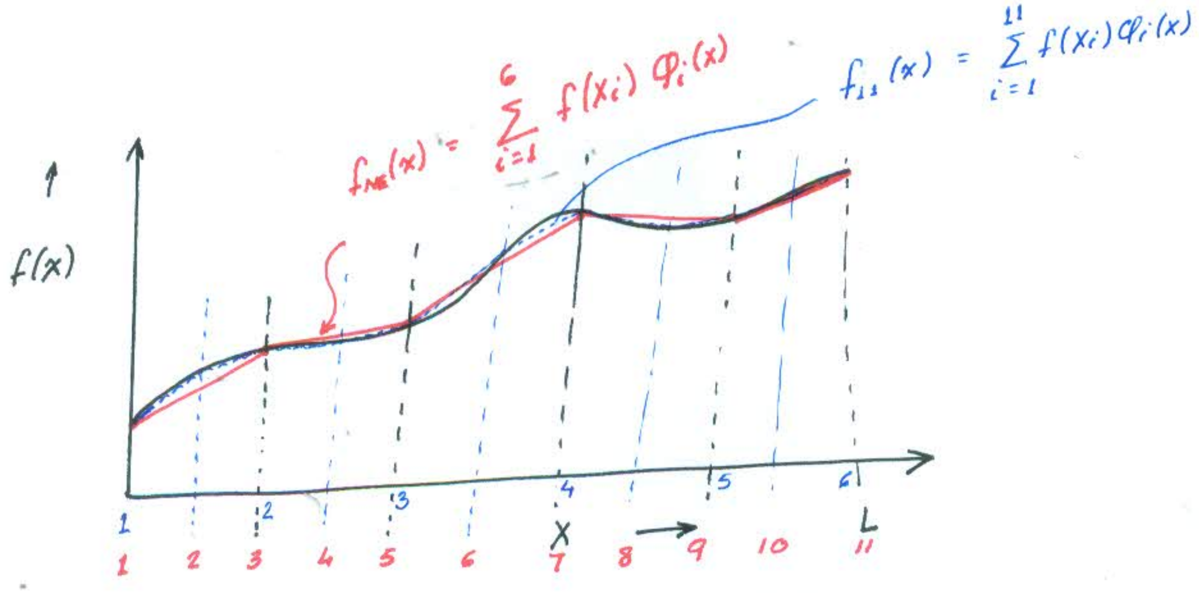
Claim: (1) $\sum_{i=1}^{NE+1} \varphi_i(x) = 1$ everywhere

(2) In each I_ℓ ; $a_0 + a_1 x = \alpha_1^\ell \varphi_\ell(x) + \alpha_2^\ell \varphi_{\ell+1}(x)$ can be represented, i.e. piecewise linear functions can be represented in terms of $\varphi_k(x)$ can be EXACTLY represented

In fact : $\alpha_1^\ell = a_0 + a_1 x_\ell$; $\alpha_2^\ell = a_0 + a_1 x_{\ell+1}$

(3) The $\varphi_k(x)$ are C^0 or continuous with jump in derivative at x_{k-1}, x_k, x_{k+1}

(4) Any ~~polyno~~ function can be approximated well by a combination of the $\{\varphi_k(x)\}_{k=1}^{NE+1}$


$f_{NE}(x) = \sum_{i=1}^{6} f(x_i)\,\varphi_i(x)$
$f_{11}(x) = \sum_{i=1}^{11} f(x_i)\varphi_i(x)$
$f(x)$
1 2 3 4 5 6
1 2 3 4 5 6 7 8 9 10 11
X
L

* - $f_6(x)$ is a six-term interpolation of $f(x)$

- $f_{11}(x)$ is a refined eleven-term interpolation of $f(x)$

$f_{11}(x)$ is "closer" to $f(x)$ than $f_6(x)$.

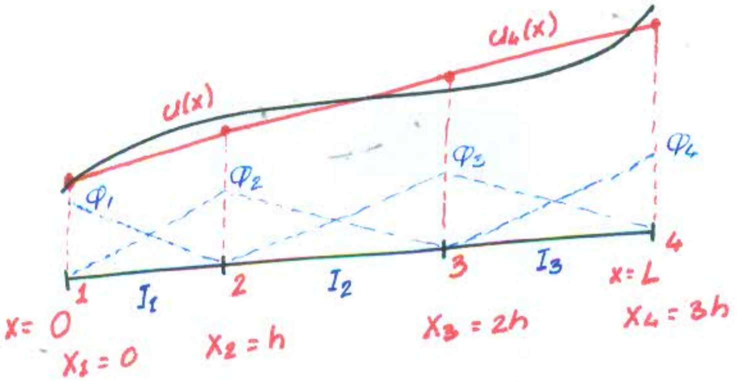
★ Thus, by taking smaller elements or more no. of elements NE, we can get a better approximation of $f(x)$. In the limit, i.e. $f(x) = \lim_{NE \to \infty} \sum_{i=1}^{NE+1} f(x_i)\varphi_i(x)$

With $x_i = (i-1)\cdot L/(NE) = (i-1)h$

$(h = L/NE)$. h → UNIFORM MESH - SIZE

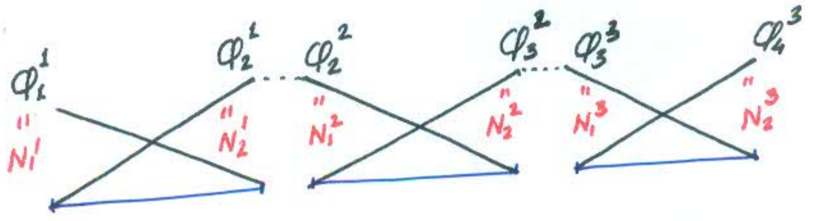
∴ We can approximate $u(x)$ by $u_h(x) = \sum_{i=1}^{NE+1} \alpha_i\,\varphi_i(x)$

. We will also choose $v(x) = v_h(x) = \sum_{i=1}^{NE+1} \beta_i \varphi_i(x)$ with β_i chosen by us.



$h = \frac{L}{3}$

$$u_4(x) = \sum_{i=1}^{4} d_i\, \varphi_i(x)$$



Element No.	Local	Global
k	1	k
	2	k+1

k

$$\varphi_k^{k}\Big|_{I_k} = \boxed{N_1^k} \longrightarrow \text{element shape function}$$

$$\varphi_{k+1}\Big|_{I_k} = \boxed{N_2^k}$$

Following our earlier derivation, we will create equation

$$B(u_n, \varphi_i) = F(\varphi_i) \qquad \text{for} \quad i = 1, 2, \cdots, (NE+1)$$

$$\Rightarrow \sum_{j=1}^{NE+1} \alpha_j\, B(\varphi_j, \varphi_i) = F(\varphi_i)$$

Note that from the figure it is easy tosee that:

$$B(\varphi_{i-1}, \varphi_i);\ B(\varphi_i, \varphi_i);\ B(\varphi_{i+1}, \varphi_i) \neq 0$$

all other $B(\varphi_j, \varphi_i) = 0$

Also

$$B(\varphi_{i-1}, \varphi_i) = B(\overbrace{\varphi_{i-1}|_{I_{i-1}}}^{N_1^{i-1}}, \overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}})$$

$$B(\varphi_{i+1}, \varphi_i) = B(\overbrace{\varphi_{i+1}|_{I_i}}^{N_2^{i}}, \overbrace{\varphi_i|_{I_i}}^{N_1^{i}})$$

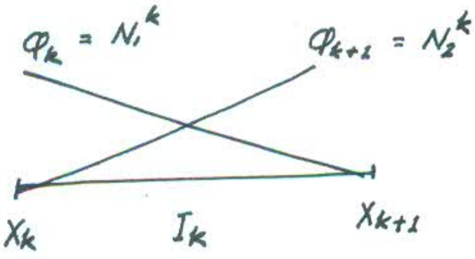
$$B(\varphi_i, \varphi_i) = \underbrace{B(\overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}}, \overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}})} + \underbrace{B(\overbrace{\varphi_i|_{I_i}}^{N_1^{i}}, \overbrace{\varphi_i|_{I_i}}^{N_1^{i}})}$$

Element wise integrals

$$F(\varphi_i) = \overbrace{F(\varphi_i|_{I_{i-1}}) + F(\varphi_i|_{I_i})}$$

ELEMENT - LEVEL INTEGRALS

$$\left.\begin{array}{l} B(u_n, \varphi_k) \\ B(u_n, \varphi_{k+1}) \\ F(\varphi_k) \\ F(\varphi_{k+1}) \end{array}\right\}$$ will have NON-ZERO contribution from I_k



$$u_n|_{I_k} = \alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1} \equiv \alpha_k N_1^k + \alpha_{k+1} N_2^k \equiv \alpha_1^k N_1^k + \alpha_2^k N_2^k$$

$\alpha_1^k \rightarrow \alpha_k$; $N_1^k \rightarrow \varphi_k$

$\alpha_2^k \rightarrow \alpha_{k+1}$; $N_2^k \rightarrow \varphi_{k+1}$

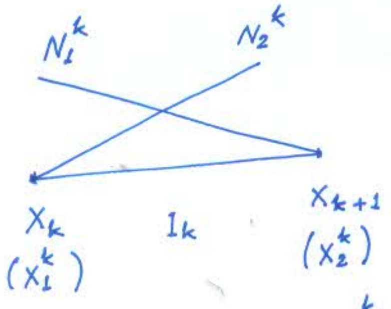
$$\therefore \quad B(\alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1}, \varphi_k) = \alpha_k \overbrace{B(\varphi_k, \varphi_k)}^{B_k(1,1)} + \alpha_{k+1} \overbrace{B(\varphi_{k+1}, \varphi_k)}^{B_k(2,1) = B_k(1,2)}$$

$$B(\alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1}, \varphi_{k+1}) = \alpha_k \underbrace{B(\varphi_k, \varphi_{k+1})} + \alpha_{k+1} \underbrace{B(\varphi_{k+1}, \varphi_{k+1})}_{B_k(2,2)}$$

Thus, we need to obtain $B_k(N_i^k, N_j^k)$ $i,j = 1,2$ ← Element stiffness matrix

and $F_k(N_i^k)$ ← Element Load Vector

Element Calculation (Element I_k)



$$N_1^k(x) = \frac{(x_2^k - x)}{(x_2^k - x_1^k)} = \frac{(x_2^k - x)}{h_k}$$

$$N_2^k(x) = \frac{(x - x_1^k)}{(x_2^k - x_1^k)} = \frac{(x - x_1^k)}{h_k}$$

$$\Rightarrow \frac{dN_1^k}{dx} = -\frac{1}{h_k} \; ; \; \frac{dN_2^k}{dx} = \frac{1}{h_k}$$

$$K_{ij}^k = B_k(N_i^k, N_j^k) = \int_{x_1^k}^{x_2^k} \left(EA \frac{dN_i^k}{dx} \frac{dN_j^k}{dx} + k_0 N_i^k N_j^k\right) dx$$

$$\Rightarrow K_{11}^k = \int_{x_1^k}^{x_2^k} \left(EA\left(-\frac{1}{h_k}\right)^2 + k_0 \frac{(x_2^k - x)^2}{h_k^2}\right) dx = \frac{EA}{h_k} + \frac{k_0}{3h_k^2}(h_k)^3$$

$$= \frac{EA}{h_k} + \frac{k_0 h_k}{3}$$

$$K_{22}^k = \int_{x_1^k}^{x_2^k} \left[EA(1/h_k)^2 + k_0 \frac{(x - x_1^k)^2}{h_k^2}\right] dx = \frac{EA}{h_k} + \frac{k_0}{3h_k^2} h_k^3$$

$$= \frac{EA}{h_k} + \frac{k_0 h_k}{3}$$

$$K_{12}^k = K_{21}^k = \int_{x_1^k}^{x_2^k} \left[EA(-1/h_k)^2 + \frac{k_0}{h_k^2}(x - x_1^k)(x_2^k - x)\right] dx$$

$$= -\frac{EA}{h_k} + k_0 \frac{h_k}{6}$$

$$\Rightarrow [K^k] = \begin{bmatrix} \left(\frac{EA}{h_k} + \frac{k_0 h_k}{3}\right) & \left(-\frac{EA}{h_k} + \frac{k_0 h_k}{6}\right) \\ \left(-\frac{EA}{h_k} + \frac{k_0 h_k}{6}\right) & \left(\frac{EA}{h_k} + \frac{k_0 h_k}{3}\right) \end{bmatrix}$$

$$F_1^k = \int_{x_1^k}^{x_2^k} f_0 N_1^k \, dx = \frac{f_0 \cdot h_k}{2} \; ; \; F_2^k = \int_{x_1^k}^{x_2^k} f_0 N_2^k \, dx = \frac{f_0 h_k}{2}$$

$$\{F^k\} = \begin{Bmatrix} f_0 h_k/2 \\ f_0 h_k/2 \end{Bmatrix}$$